BOUNDING COHOMOLOGY OF PROJECTIVE SCHEMES

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Preface

One of the most fascinating aspects of Projective Algebraic Geometry is the interplay between discrete and continuous data of a projective algebraic variety, or more generally of a closed subscheme X of a given projective space $\mathbb{P}^r_K$ over some (algebraically closed) field K. In particular, one approach to projective schemes consists in studying classes of closed subschemes $X \subseteq \mathbb{P}^r_K = \mathrm{Proj}(K[X_0, X_1, \ldots, X_r])$ with a given "discrete skeleton".

A most prominent occurence of this method comes up in the theory of Hilbert schemes, where the class of all closed subschemes $X \subseteq \mathbb{P}^r_K$ is sliced up into classes of subschemes X having a given Hilbert-Serre polynomial $p = P_X = P_{\mathcal{O}_X}$. In this particular situation, the objects in each of the slices are parametrized by a projective scheme, the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$. Here the discrete skeleton is given by the Hilbert-Serre polynomial p and the continuous data are encoded in the corresponding Hilbert scheme.

Another occurence of the same principle is to use Betti numbers

$$\beta_{i,j}(I_X) := \dim_K\big(\mathrm{Tor}_i^{K[X_0,X_1\ldots,X_r]}(K, I_X)_{j+i}\big)$$

of X (that is of the homogeneous vanishing ideal $I_X \subseteq K[X_1, X_2, \ldots, X_r]$) to define the discrete skeleton and to study all closed subschemes $X \subseteq \mathbb{P}^r_K$ with a given Betti table

$$\beta(X) := \big(\beta_{i,j}(I_X)\big)_{(i,j)\in\{0,1,\ldots,r\}\times\mathbb{N}}.$$

As the Betti table of $X \subseteq \mathbb{P}^r_K$ determines the Hilbert-Serre polynomial $p = P_X = P_{\mathcal{O}_X}$ of X, the "slices" now are are "thinner" than in the previous case and form indeed locally closed strata of the corresponding Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$. Moreover in this situation the "thick slice“ $\mathrm{Hilb}^p_{\mathbb{P}^r}$ contains only finitely many of the "thin slices" given by fixing the Betti table $\beta(X)$ of X.

Finally, one also could use sheaf cohomology to define the discrete skeleton. In the previously described situation, this would mean to slice up the scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$ into strata consisting of all closed subschemes $X \subseteq \mathbb{P}^r_K$ (with Hilbert-Serre polynomial p, obviously) and given "cohomology table"

$$h_{X,\mathcal{O}_X} = h_{\mathcal{O}_X} := \big(h^i(X, \mathcal{O}_X(n))\big)_{(i,n)\in\{0,1,\ldots,r\}\times\mathbb{Z}}.$$

This point of view is supported by the fact that imposing an arbitrary lower bound on the cohomology table $h_{\mathcal{O}_X}$ always leaves us with a locally closed and (rationally) connected stratum in $\mathrm{Hilb}^p_{\mathbb{P}^r}$ (see [Fu]). Moreover, in this case, the "thick slice" $\mathrm{Hilb}^p_{\mathbb{P}^r}$ contains only finitely many of the "thin slices" given by fixing (the Hilbert-Serre polynomial $p = P_X = P\mathcal{O}_X$, of course) and the cohomology table $h_{\mathcal{O}_X}$ of X.

One finally could consider a more general situation, which exceeds the framework of Hilbert schemes and just slice up the class $\mathcal{S}^t$ of all pairs $(X, \mathcal{F})$ in which X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of

$\mathcal{O}_X$-modules with $\dim(\mathcal{F}) = t$ by means of the "cohomology table"

$$h_{X,\mathcal{F}} = h_{\mathcal{F}} := \left(h^i(X, \mathcal{F}(n))\right)_{(i,n)\in\{0,1,\ldots,t\}\times\mathbb{Z}}$$

of X with respect to the coherent sheaf of $\mathcal{O}$-modules $\mathcal{F}$. Clearly, in this situation one cannot expect that the class $\mathcal{S}^t$ splits up into finitely many ot the "slices" given by fixing the cohomology the table $h_{\mathcal{F}}$. Nevertheless in the interst of effectivity one should also would like to know in this more general setting, whether the single slices are not "too thin" so that the family of slices is "not too large".

A way of understanding this "thickness problem for slices obtained by fixing cohomology tables" would be to describe all possible cohomology tables $h_{\mathcal{F}}$ if $(X, \mathcal{F})$ runs through the full class $\mathcal{S}^t$. A more realistic approach would be to prove some "finiteness results" which hint, that the variety of occuring slices is not too large. This is actually one of the basic aims of these lectures. In particular, we shall show that prescribing the entries along certain finite patterns $\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ - which we call quasi-diagonal sets - leaves us with only finitely many possible cohomology tables. Quasi-diagonal sets are of the shape $\{(i, n_i) \mid i = 0, 1, \ldots, t\}$ with $n_t < n_{t-1} < \ldots < n_1 < n_0$.

The main ingredient for the proof of this finiteness result is a bounding result on the "cohomological postulation numbers". To be more precise, let $i \in \mathbb{N}_0$. Then the i-th cohomological postulation number $\nu^i_{\mathcal{F}}$ of $\mathcal{F}$ is the ultimate place $n \in \mathbb{Z}$ at which $h^i(X, \mathcal{F}(n))$ does not take the same value as the "i-th cohomological Serre polynomial" $p^i_{\mathcal{F}}$ of $\mathcal{F}$ which is defined by the property that $h^i(X, \mathcal{F}(m)) = p^i_{\mathcal{F}}(m)$ for all $m \ll 0$. Our bounding result says that the numbers $\nu^i_{\mathcal{F}}$ find a lower bound in terms of the "cohomology diagonal"

$$\Delta_{\mathcal{F}} := \left(h^i(X, \mathcal{F}(-i)\right)_{i=0,1,\ldots,t} \quad .$$

This bounding result on its turn will follow from another result, which says that the "Castelnuovo-Mumford regularity" $\mathrm{reg}(K^i(M))$ of the "i-th deficiency module" $K^i(M)$ of a finitely generated graded module over a Noetherian homogeneous K-algebra is bounded in terms of the "beginning" $\mathrm{beg}(M) := \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}$ and the "geometric cohomology diagonal" $\Delta_{\widetilde{M}}$ of M, that is the cohomology diagonal of the coherent sheaf $\widetilde{M}$ of $\mathcal{O}_{\mathrm{Proj}(R)}$-modules induced by M.

Besides of these main results, we shall also discuss a number of related subjects and their history. Already here we recommend the lecture notes [Br8] for a more complete and fairly self-contained presentation of the subject. The prerequisites which are needed to follow the notes [Br8] are contained in [Br-Fu-Ro] or in [Br-Sh1].

The present notes are divided into six sections. In Section 1 we introduce the basic notions and concepts which we shall need later. We freely use the necessary background from Graded Local Cohomology Theory and also from Sheaf Cohomology Theory over Projective Schemes. In particular we define the notion of "subclass $\mathcal{D} \subseteq \mathcal{S}^t$ of finite cohomology (on a subset $\mathbb{S} \subseteq \{0, 1, 2 \ldots, t\} \times \mathbb{Z}$)", which will be of fundamental meaning in these lectures. We also present examples of subclasses $\mathcal{D} \subseteq \mathcal{S}^t$ which are of finite cohomology and subclasses of $\mathcal{S}^t$ which are not.

In Section 2 we are interested in supporting degrees of cohomology and in the related notion of "cohomological pattern" of a pair $(X, \mathcal{F}) \in \mathcal{S}^t$. This pattern is the set of pairs $(i, n) \in \{0, 1, \ldots, t\} \times \mathbb{Z}$ for which the entry $h^i(X, \mathcal{F}(n))$ of the cohomology table $h_{\mathcal{F}}$ at the place (i, n) does not vanish. Without proof we state a combinatorial chracterization of these cohomological patterns. In this section, we also recall the notion of "Castelnuovo-Mumford regularity" and its basic properties. In addition, we give a brief (and partially historic) account on the "Vanishing Theorem of Severi-Enriques-Zariski-Serre".

In Section 3 we introduce the "deficiency modules" $K^i(M)$ of a finitely generated graded module M over a Noetherian homogeneous K-algebra R. As we restrict ourselves to work over Noetherian homogeneous algebras over a fields, we can do this in a "narrow-gauge" manner. This has the advantage to encode a fortiori the graded form of Grothendieck's Local Duality Theorem or - equivalently - the Serre Duality Theorem. We first calculate the deficiency modules of a polynomial ring over a field and then prove the basic properties of such modules in general. In a next step we use deficiency modules to introduce the notion of "cohomological Hilbert polynomial", of "cohomological Serre polynomial" and of "cohomological postulation number". We also briefly consider the special case of "canonical modules" and prove that these latter satisfy (a weak form of) the "second Serre property S_2".

In Section 4 we use the tools developed in Section 3 to establish a number of bounding results for the Castelnuovo-Mumford regularity of finitely generated graded modules over Noetherian homogeneous algebras over a field. The most prominent of these is the main result we already announced above: an upper bound on the regularity of the deficiency modules $K^i(M)$ of a finitely generated graded R-module M over a Noetherian homogeneous K-algebra R in terms of the beginning and the geometric cohomology diagonal of M. This result has in fact a number of further applications, which are beyond the reach of this course and for which the intersted reader is recommended to consult [Br-Ja-Li1].

In Section 5 we first prove the previously anounced bounding result on the cohomological postulation numbers $\nu^i_{\mathcal{F}}$, where $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules and X is a projective scheme over some field K. As an application we show that a class $\mathcal{D} \subseteq \mathcal{S}^t$ which is of finite cohomology on an arbitrary quasi-diagonal subset $\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is of bounded cohomology at all. Our final conclusion is, that a subset $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is a bounding set for cohomology (which means that each subclass $\mathcal{D} \subseteq \mathcal{S}^t$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all) if and only if it contains a quasi-diagonal set Σ. We conclude this section with a number of remarks and open questions. To readers, who aim to see a complete account and further consequences of the results of Section 5, we recommend to consult [Br-Ja-Li2].

In Section 6 we give a sample of bibliographical hints, based on the bibliography of [Br8].

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1. Introduction

These lectures focus on a basic question of projective algebraic geometry, namely:

1.1. **Question.** *What bounds cohomology of a projective scheme?*

We first shall make more precise this question. To do so, we have to introduce a few notions. We shall define the basic concepts and formulate our main results primarily in the geometric language of schemes and coherent sheaves. But we also shall recall the necessary tools from local cohomology theory which allow to translate our results to the purely algebraic language of graded rings and modules - and vice versa.

Our main results are contained in our joint work with Hellus [Br-He] and Jahangiri-Linh [Br-Ja-Li1],[Br-Ja-Li2] and in the lecture notes [Br8]. To simplify matters we shall content ourselves in this course to consider projective schemes over fields instead of projective schemes over local Artinian rings. A complete and fairly selfcontained exposition of our results is given in [Br8]. As this set of lecture notes is available online under

www.math.uzh.ch/brodmann (click: my site at the department / click: Publikationen)

we allow ourselves to quote this source repeatedely. For those readers, who wish to consult a selfcontained introduction to the foundations of local cohomology and sheaf cohomology over projective schemes, we recommend [Br-Fu-Ro], which is also available under the previous URL. To readers, who like to start from a more extended background we recommend to consult [Br-Sh1] and Chapters II and III of [H1]. To illustrate the significance of the whole subject we also shall present a few classical results and discuss their relationship with the topic of these lectures. The notion of Castelnuovo-Mumford regularity plays an important rôle in our lectures and so they are related to those given by L.T.Hoa.

As announced above, we now shall make precise the question asked at the beginning of our exposition. Throughout, we use $\mathbb{N}$ to denote the set of strictly positive integers and $\mathbb{N}_0$ to denote the set of non-negative integers. We first introduce the basic notations we shall use in these lectures

1.2. **Notation.** *(Two Basic Classes)* A) Let $d \in \mathbb{N}$. By $\mathcal{M}^d$ we shall denote the class of all pairs (R, M) such that

$$R = K \oplus R_1 \oplus R_2 \oplus \ldots = K[R_1]$$

is a Noetherian homogeneous K-algebra over some field K (hence R is $\mathbb{Z}$-graded and generated as a K-algebra by finitely many alements of degree 1) and

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

is a finitely generated graded R-module whose Krull dimension satisfies $\dim_R(M) = d$.

B) Let $t \in \mathbb{N}_0$. By $\mathcal{S}^t$ we shall denote the class of all pairs $(X, \mathcal{F})$ such that X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules whose Krull dimension satisfies

$$\dim(\mathcal{F}) := \dim\big(\mathrm{Supp}(\mathcal{F})\big) = \sup\{\dim_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in X\} = t.$$

In the present course, we prefer to formulate our final results in terms of sheaf cohomology over projective schemes, hence in the framework of the classes introduced in part B) of (1.2). Nevertheless, as many readers may prefer to think on the whole subject in algebraic terms, hence on the base of the classes introduced in part A) of (1.2), we shall introduce our basic notions in both formalisms. We consider this as a way of encouraging "Pure Algebraists" to through a glance to the rich and appealing geometric phenomena related to the algebraic formalisms we shall use. Moreover, the subjects we are speaking about have there roots in Algebraic Geometry.

1.3. **Remark.** A) *(Relating the Two Basic Classes)* Let $t \in \mathbb{N}_0$. Then by the well known relation between graded modules over homogeneous rings and coherent sheaves over projective schemes we may write

$$\mathcal{S}^t = \{(\mathrm{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{M}^{t+1}),$$

where $\mathrm{Proj}(R)$ denotes the projective scheme induced by the Noetherian homogeneous ring R and $\widetilde{M}$ denotes the coherent sheaf of $\mathcal{O}_{\mathrm{Proj}(R)}$-modules induced by the finitely generated graded R-module M (see [H1] Chapters II and III, [Br-Sh1] Chapter 20, or [Br-Fu-Ro] Sections 11 and 12). It should be noted, that the assignement

$$(R, M) \mapsto (\mathrm{Proj}(R), \widetilde{M})$$

does not define a bijection between the classes $\mathcal{M}^{t+1}$ and $\mathcal{S}^t$. Indeed, for two finitely graded R-modules $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $N = \bigoplus_{n\in\mathbb{Z}} N_n$ with $M \subseteq N$ one has $\widetilde{M} = \widetilde{N}$ if and only if $M_n = N_n$ for all $n \gg 0$.

B) *(Shifting and Twisting)* Keep the above notations and hypotheses. Let $(R, M) \in \mathcal{M}^d$, let $X := \mathrm{Proj}(R)$ and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf of $\mathcal{O}_X$-modules induced by M, so that $(X, \mathcal{F}) \in \mathcal{S}^{d-1}$. For each $n \in \mathbb{Z}$ let $M(n)$ denote the *n-th shift of* M and let $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ (with $\mathcal{O}_X(n) := \widetilde{R(n)}$) be the *$n$-th twist of* $\mathcal{F}$. Then we have (see [H1])

$$\mathcal{F}(n) = \widetilde{M(n)}, \quad \forall n \in \mathbb{Z}.$$

In particular, we can say that the classes $\mathcal{M}^d$ and $\mathcal{S}^t$ are closed under shifting respectively twisting:

a) If $(R, M) \in \mathcal{M}^d$, then $(R, M(n)) \in \mathcal{M}^d$ for all $n \in \mathbb{Z}$.

b) If $(X, \mathcal{F}) \in \mathcal{S}^t$, then $(X, \mathcal{F}(n)) \in \mathcal{S}^t$ for all $n \in \mathbb{Z}$.

1.4. **Notation and Reminder.** A) *(Local Cohomology and Algebraic Cohomological Hilbert Functions)* Let $d \in \mathbb{N}$ and let $(R, M) \in \mathcal{M}^d$, with $R = K \oplus R_1 \oplus R_2 \ldots$ a Noetherian homogeneous algebra over the field K. Througout in this situation we write

$$R_+ := \bigoplus_{n \in \mathbb{N}} R_n$$

for the *irrelevant ideal of* R. Moreover, for each integer $i \in \mathbb{N}_0$ let

$$H^i_{R_+}(M) = \big(\mathcal{R}^i \Gamma_{R_+}\big)(M) = \lim_{\underset{n}{\longrightarrow}} \mathrm{Ext}^i_R(R/(R_+)^n, M)$$

denote the *i-th local cohomology module of M with respect to R_+*, that is the i-th right derived of the *R_+-torsion-functor* $\Gamma_{R_+}(\bullet) = \lim_{\underset{n}{\longrightarrow}} \mathrm{Hom}_R(R/(R_+)^n, \bullet)$ with respect to R_+ evaluated at the object M. Keep in mind the well known fact that the R-modules $H^i_{R_+}(M)$ carry a natural grading and that for the corresponding graded components we have (see [Br-Sh1], [Br-Fu-Ro] or [Br8])

a) $h^i_M(n) := \dim_K(H^i_{R_+}(M)_n) < \infty$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
b) $h^i_M(n) = 0$ for all $i \in \mathbb{N}_0$ and all $n \gg 0$.
c) $h^i_M(n) = 0$ for all $i > d$ and all $n \in \mathbb{Z}$.
d) $h^d_M(n) \neq 0$ for all $n \ll 0$.

In particular, for each $i \in \mathbb{N}_0$, we may define the *i-th algebraic cohomological Hilbert function of M*, that is the right-vanishing function

$$h^i_M : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto h^i_M(n), \quad \forall n \in \mathbb{Z}.$$

B) *(Ideal Transforms and Geometric Cohomological Hilbert Functions)* Keep the notations and hypotheses of part A). We consider the *i-th R_+-transform of M*, that is the R-module

$$D^i_{R_+}(M) := \big(\mathcal{R}^i D_{R_+}\big)(M) = \lim_{\underset{n}{\longrightarrow}} \mathrm{Ext}^i_R((R_+)^n, M),$$

obtained by evaluating the i-th right derived of the *R_+-transform functor* $D_{R_+}(\bullet) = \lim_{\underset{n}{\longrightarrow}} \mathrm{Hom}_R((R_+)^n, \bullet)$ at the object M. Keep in mind, that the R-modules $D^i_{R_+}(M)$ carry a natural grading and morevoer (see [Br-Sh1])

a) There is a natural short exact sequence of graded R-modules

$$0 \to H^0_{R_+}(M) \to M \to D^0_{R_+}(M) \to H^1_{R_+}(M) \to 0.$$

b) For all $i \in \mathbb{N}$ there is a natural isomorphism of graded R-modules

$$D^i_{R_+}(M) \cong H^{i+1}_{R_+}(M).$$

In particular it follows from statements a)-d) of part A) that

c) $d^i_M(n) := \dim_K(D^i_{R_+}(M)_n) < \infty$, for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
d) $d^0_M(n) = \dim_K(M_n) + h^1_M(n) - h^0_M(n)$ for all $n \in \mathbb{Z}$.
e) $d^0_M(n) = \dim_K(M_n)$ for all $n \gg 0$.
f) $d^i_M(n) = h^{i+1}_M(n)$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$.

g) $d^i_M(n) = 0$ for all $i \in \mathbb{N}$ and all $n \gg 0$.
h) $d^i_M(n) = 0$ for all $i \geq d$ and all $n \in \mathbb{Z}$.
i) $d^{d-1}_M(n) \neq 0$ for all $n \ll 0$.

Now, for each $i \in \mathbb{N}_0$ we may define the *i-th geometric cohomological Hilbert function of M* as the function

$$d^i_M : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto d^i_M(n), \quad \forall n \in \mathbb{Z}.$$

C) *(Behaviour in Short Exact Sequences)* Let K be a field, let R be a Noetherian homogeneous K-algebra and let

$$\mathbb{S} : 0 \to L \to M \to N \to 0$$

be an exact sequence of finitely generated graded R-modules. Then, on use of the right derived exact sequences of the functors $\Gamma_{R_+}(\bullet)$ and $D_{R_+}(\bullet)$ one obtains the following inequalities for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (and observing the notational convention that $h^j_M = d^j_M = 0$ for all integers $j < 0$):

a) $h^i_M(n) \leq h^i_L(n) + h^i_N(n)$.
b) $h^i_L(n) \leq h^i_M(n) + h^{i-1}_N(n)$.
c) $h^i_N(n) \leq h^i_M(n) + h^{i+1}_L(n)$.
d) $d^i_M(n) \leq d^i_L(n) + d^i_N(n)$.
e) $d^i_L(n) \leq d^i_M(n) + d^{i-1}_N(n)$.
f) $d^i_N(n) \leq d^i_M(n) + d^{i+1}_L(n)$.

Now, we aim to link the above algebraic concepts to sheaf theory.

1.5. **Notation and Reminder.** A) *(Serre Cohomology of Projective Schemes with Coefficients in Coherent Sheaves)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$, such that X is a projective scheme over the field K. For each $i \in \mathbb{N}_0$ and each $n \in \mathbb{Z}$ let

$$H^i(X, \mathcal{F}(n)) := \big(\mathcal{R}^i\Gamma(X, \bullet)\big)(\mathcal{F}(n))$$

denote the *i-th Serre (or sheaf) cohomology group of (X with coefficients in) the n-th twist $\mathcal{F}(n)$ of $\mathcal{F}$*, that is the i-th right derived of the functor $\Gamma(X, \bullet)$ of taking global sections, evaluated at the object $\mathcal{F}(n)$. Keep in mind, that the cohomology groups $H^i(X, \mathcal{F}(n))$ all carry a natural structure of K-vector space.

B) *(The Serre-Grothendieck Correspondence and Cohomological Hilbert Functions of Coherent Sheaves)* Let the notations and hypotheses be as in part A). Then, according to (1.3) A) there is a (not necessarily unique) pair $(R, M) \in \mathcal{M}^{t+1}$ such that

$$(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$$

and in this situation the *Serre-Grothendieck Correspondence* (see [Br-Sh1], Chapter 20) yields that for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ there is an isomorphism of K-vector spaces

$$H^i(X, \mathcal{F}(n)) \cong D^i_{R_+}(M)_n.$$

In particular we have

$$h^i_{\mathcal{F}}(n) = h^i(X, \mathcal{F}(n)) := \dim_K\big(H^i(X, \mathcal{F}(n))\big) = d^i_M(n), \quad \forall i \in \mathbb{N}_0, \forall n \in \mathbb{Z}.$$

This allows to define for each $i \in \mathbb{N}_0$ the *i-th cohomological Hilbert function of (X with coefficients in) $\mathcal{F}$*, that is the function

$$h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto h^i_{\mathcal{F}}(n) = h^i(X, \mathcal{F}(n)), \quad \forall n \in \mathbb{Z}.$$

C) *(First Properties of Cohomological Hilbert Functions of Coherent Sheaves)* Let the notations and hypotheses be as in parts A) and B). In particular, let $(R, M) \in \mathcal{M}^{t+1}$ with $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. Then by the observations made in part B) we have

$$h^i_{\mathcal{F}} = d^i_M, \quad \forall i \in \mathbb{N}_0.$$

This should explain why we called the functions d^i_M the *geometric* cohomological Hilbert functions of M: they actually describe the geometric object associated to the algebraic object M. By statements g), h) and i) of (1.4) B) we now respectively obtain

a) $h^i_{\mathcal{F}}(n) = 0$ for all $n \gg 0$.
b) $h^i_{\mathcal{F}}(n) = 0$ for all $i > t = \dim(\mathcal{F})$ and all $n \in \mathbb{Z}$.
c) $h^t_{\mathcal{F}}(n) \neq 0$ for all $n \ll 0$.

D) *(Relating Cohomological Hilbert Functions of Sheaves and Modules)* Let the notations and hypotheses be as in part C). Then, the equalities observed at the beginning of part C) together with with the relations 1.4 d),f) imply that

a) $h^i_{\mathcal{F}}(n) = h^{i+1}_M(n)$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$.
b) $h^0_{\mathcal{F}}(n) = h^1_M(n)$ for all $n \ll 0$.

In order to collect all cohomological Hilbert functions of a pair in $\mathcal{M}^d$ respectively in $\mathcal{S}^t$, we give the following definition.

1.6. **Definition and Remark.** A) *(Cohomology Tables of Graded Modules)* Let $d \in \mathbb{N}$ and let $(R, M) \in \mathcal{M}^d$. We define the *algebraic cohomology table of* M as the family of non-negative integers

$$h_M := \big(h^i_M(n)\big)_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \quad .$$

The *geometric cohomology table of* M is defined as the family of non-negative integers

$$d_M := \big(d^i_M(n)\big)_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \quad .$$

For a subset $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ we also aim to consider the *restricted cohomology tables of* M, that is the restricted families of non-negative integers

$$h_M \restriction_{\mathbb{S}} := \big(h^i_M(n)\big)_{(i,n) \in \mathbb{S}} \quad ,$$

$$d_M \restriction_{\mathbb{S}} := \left(d^i_M(n)\right)_{(i,n)\in\mathbb{S}} \quad .$$

B) *(Cohomology Tables of Coherent Sheaves)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. We define the *cohomology table of the (scheme X with coefficients in the) coherent sheaf $\mathcal{F}$* as the family of non-negative integers

$$h^i_{\mathcal{F}} := \left(h^i_{\mathcal{F}}(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad .$$

Correspondingly, for any subset $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$, we define the *restricted cohomology table of $\mathcal{F}$* as the restricted family of non-negative integers

$$h_{\mathcal{F}} \restriction_{\mathbb{S}} := \left(h^i_{\mathcal{F}}(n)\right)_{(i,n)\in\mathbb{S}} \quad .$$

C) *(Identifying Cohomology Tables of Sheaves and of Modules)* Let the notations be as in part B), and let $(R, M) \in \mathcal{M}^{t+1}$ with $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$. Then, it follows from (1.5) C) that

$$h_{\mathcal{F}} \restriction_{\mathbb{S}} = d_M \restriction_{\mathbb{S}} .$$

This tells us, that instead of (restricted) cohomology tables of coherent sheaves we may content ourselves to consider (restricted) geometric cohomology tables of graded modules-and vice versa.

1.7. **Definition and Remark.** A) *(Classes of Finite Cohomology: the Case of Modules)* Let $d \in \mathbb{N}$ and let $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$. We say that a sublass $\mathcal{C} \subseteq \mathcal{M}^d$ is *of finite cohomology on* $\mathbb{S}$ if the set of families

$$\{d_M \restriction_{\mathbb{S}} \ \mid (R, M) \in \mathcal{C}\} = \{\left(d^i_M(n)\right)_{(i,n)\in\mathbb{S}} \mid (R, M) \in \mathcal{C}\}$$

is finite. Clearly in view of 1.4 B)g) it suffices to consider this conditions only for sets

$$\mathbb{S} \subseteq \{0, 1, \ldots, d-1\} \times \mathbb{Z}.$$

We say that the class $\mathcal{C} \subseteq \mathcal{M}^d$ *is of finite cohomology (at all)* if it is of finite cohomology on the set $\{0, 1, \ldots, d-1\} \times \mathbb{Z}$ or, equivalently, on an arbitrary set $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ containing the former.

B) *(Classes of Finite Cohomology: the Case of Sheaves)* Let $t \in \mathbb{N}_0$ and let $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$. We say that a subclass $\mathcal{D} \subseteq \mathcal{S}^t$ is *of finite cohomology on* $\mathbb{S}$ if the set of families

$$\{h_{\mathcal{F}} \restriction_{\mathbb{S}} \ \mid (X, \mathcal{F}) \in \mathcal{D}\} = \{\left(h^i_{\mathcal{F}}(n)\right)_{(i,n)\in\mathbb{S}} \mid (R, M) \in \mathcal{C}\}$$

is finite. Similarly as in part A) it suffices to consider sets $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$, and also similarly as in part A) we say that the class $\mathcal{D} \subseteq \mathcal{S}^t$ *is of finite cohomology (at all)* if it is of finite cohomology on the set $\{0, 1, \ldots, t\} \times \mathbb{Z}$ or, again equivalently, on an arbitrary set $\mathbb{S} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ containing the former.

C) *(Relating the Two Notions of Part A) and B))* Let t, $\mathbb{S}$, and $\mathcal{D} \subseteq \mathcal{S}^t$ be as in part B). Then, according to (1.3) A) there is a sublass $\mathcal{C} \subseteq \mathcal{M}^{t+1}$ such that

$$\mathcal{D} = \{(\mathrm{Proj}(R), \widetilde{M}) \mid (R, M) \in \mathcal{C}\}.$$

In this situation it follows from (1.6) C) that $\mathcal{D}$ is of finite cohomology on $\mathcal{S}$ if and only if $\mathcal{D}$ is. So, the study of classes of finite cohomology for modules and sheaves are equivalent features.

We new recall a few basic facts on *Hilbert polynomials* and *characteristic functions*, which we shall frequently use during these lectures.

1.8. **Reminder and Remark.** A) *(Hilbert Functions and Postulation Numbers of Graded Modules)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated R-module of dimension $d \in \mathbb{N}_0 \cup \{-\infty\}$. In this situation we denote the *Hilbert polynomial* of M by P_M, so that

a) $P_M \in \mathbb{Q}[X]$ with $\deg(P_M) = d - 1$ if $d > 0$ and $P_M = 0$ if $d \leq 0$.
b) $\dim_K(M_n) = P_M(n)$ for all $n \gg 0$.

In view of statement b) we may define the *postulation number of the graded R-module M* by

$$P(M) := \sup\{n \in \mathbb{Z} \mid \dim_K(M_n) \neq P_M(n)\} \in \mathbb{Z} \cup \{-\infty\}.$$

B) *(Characteristic Functions of Graded Modules)* Let R and M be as above. The *characteristic function* of M is defined by

$$\chi_M : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \chi_M(n) := \sum_{i=0}^{d-1}(-1)^i d_M^i(n) = \sum_{i\in\mathbb{N}_0}(-1)^i d_M^i(n), \quad \forall n \in \mathbb{Z}.$$

Clearly, by the observations made in (1.4) B) d),f) for all $n \in \mathbb{Z}$ we may write

a) $\chi_M(n) = \dim_K(M_n) - \sum_{i=0}^{d}(-1)^i h_M^i(n) = dim_K(M_n) - \sum_{i\in\mathbb{N}_0}(-1)^i h_M^i(n).$

A must important fact is the so called *(Algebraic) Serre Formula* which relates the characteristic function and the Hilbert polynomial of a finitely generated graded R-module (see [Br-Fu-Ro](9.17),(9.18) or [Br-Sh1] (17.1.6) for example):

b) $P_M(n) = \chi_M(n)$ for all $n \in \mathbb{Z}$.

As an easy consequence of this one gets the following estimate for the postulation number of the graded R-module M:

c) $P(M) \leq \sup\{n \in \mathbb{Z} \mid \exists i \in \mathbb{N}_0 : h_M^i(n) \neq 0\}$.

C) *(Hilbert-Serre Polynomials and Characteristic Functions of Sheaves)* Let X be a projective scheme over some field K and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) = t$. Then we may define the *characteristic function* *of $\mathcal{F}$* by

$$\chi_{\mathcal{F}} : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \chi_{\mathcal{F}}(n) := \sum_{i=0}^{t} (-1)^i h^i_{\mathcal{F}}(n) = \sum_{i \in \mathbb{N}_0} (-1)^i h^i_{\mathcal{F}}(n), \quad \forall n \in \mathbb{Z}.$$

We now find R and M as in part A) such that $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$. In this situation it follows from (1.5 C) that

$$\chi_{\mathcal{F}} = \chi_M.$$

Hence by the Algebraic Serre Equality and by statement B)b) there is a unique polynomial $P_{\mathcal{F}} \in \mathbb{Q}[X]$ (namely $P_{\mathcal{F}} := P_M$) such that

a) $\deg(P_{\mathcal{F}}) = \dim(\mathcal{F})$,
b) $\chi_{\mathcal{F}}(n) = P_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$,
c) $P_{\mathcal{F}}(n) = h^0(X, \mathcal{F}(n)) = h^0_{\mathcal{F}}(n)$ for all $n \in \mathbb{Z}$ such that $h^i_{\mathcal{F}}(n) = 0$ for all $i \in \mathbb{N}$.

This polynomial is called the *Hilbert-Serre polynomial of $\mathcal{F}$*. Now, again, we may define the *postulation number of $\mathcal{F}$* as

$$P(\mathcal{F}) := \sup\{n \in \mathbb{Z} \mid h^0(X, \mathcal{F}(n)) \neq P_{\mathcal{F}}(n)\} \in \mathbb{Z} \cup \{-\infty\}$$

and similarly as in statement B)c) we get

d) $P(\mathcal{F}) < \sup\{n \in \mathbb{Z} \mid \exists i \in \mathbb{N} : h^i_{\mathcal{F}}(n) \neq 0\}$.

One of our principal aims is to prove finiteness results for classes $\mathcal{D} \subseteq \mathcal{S}^t$ or, equivalently, for classes $\mathcal{C} \subseteq \mathcal{M}^d$. The following example can be considered as being classical. In geometric terms it says that the pairs $(X, \mathcal{O}_X)$ in which $X \subseteq \mathbb{P}^r_K$ runs through all closed subschemes with a given Hilbert polynomial p (hence the pairs $(X, \mathcal{O}_X) \in \mathcal{S}^{\deg(p)}$ parametrized by the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$)) form a class of finite cohomology.

1.9. **Example.** Let $r \in \mathbb{N}$, let $p \in \mathbb{Q}[X]$ be a polynomial with $\deg(p) < r$, let K be a field and consider the class

$$\mathcal{C} := \{(R, R) \in \mathcal{M}^{\deg(p)+1} \mid \dim_K(R_1) = r + 1; \quad P_R = p\}$$

of all pairs (R, R) in which $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ is a Noetherian homogeneous K-algebra, having Hilbert polynomial p and being generated by $r + 1$ linear forms. Then the class $\mathcal{C}$ is indeed of finite cohomology. We shall derive this fact from a much more general result which we will treat later. As the class

$$\mathcal{D} := \{(\mathrm{Proj}(R), \widetilde{R}) \mid (R, R) \in \mathcal{C}\} \subseteq \mathcal{S}^{\deg(p)}$$

is parametrized by the Hilbert scheme $\mathrm{Hilb}^p_{\mathbb{P}^r}$, we get the finiteness statement made prior to this example by (1.7) C).

1.10. **Example and Exercise.** A) Let $r > 1$ be an integer, and consider the polynomial ring

$$R = K[X_1, X_2, \ldots, X_r].$$

over the field K, furnished with its standard $\mathbb{Z}$-grading. Let

$$\mathcal{C} := \{(R, R(m)) \mid m \in \mathbb{Z}\} \subseteq \mathcal{M}^r.$$

It is well known, that the local cohomology modules of R with respect to R_+ satisfy the following requirements (see [Br-Sh1](12.4.1) or [Br8](9.6),(9.4)C)b), for example)

a) $H^i_{R_+}(R) = 0$ for all $i \neq r$,

b) $H^r_{R_+}(R)_n \cong R_{-r-n}$ for all $n \in \mathbb{Z}$.

Keeping in mind that local cohomology with respect to R_+ commutes with shifting and in view of statements d) and e) of (1.4) B) it follows with the notational convention that $\binom{k}{r-1} = 0$ for all $k < r - 1$

a) $d^0_{R(m)}(n) = \binom{r+m+n-1}{r-1}$ for all $m, n \in \mathbb{Z}$,

b) $d^{r-1}_{R(m)}(n) = \binom{-m-n-1}{r-1}$ for all $m, n \in \mathbb{Z}$,

c) $d^i_{R(m)}(n) = 0$ for all $i \neq 0, r-1$ and all $m, n \in \mathbb{Z}$.

This clearly shows that the class $\mathcal{C}$ is not of finite cohomology.

B) Keep the notations and hypotheses of part A). For each $m \in \mathbb{N}$ chose a form $f_m = R_m \setminus \{0\}$ and consider the Noetherian homogeneous graded K-algebra $R^{[m]} := R/f_m R$ with Hilbert-polynomial

$$P_{R^{[m]}} = \binom{r + X - 1}{r - 1} - \binom{r - m + X - 1}{r - 1}.$$

Observe that for each $m \in \mathbb{N}$ there is an exact sequence of graded R-modules

$$0 \to R \xrightarrow{f_m} R(m) \to R^{[m]} \to 0.$$

Use the observations made in part A) to show that the class $\{(R, R^{[m]}) \mid m \in \mathbb{N}\}$ is not of finite cohomology. Use the Base-Ring Independence Propery of Local Cohomology (see [Br-Sh1] Chapter 4, or [Br8] Section 1) to show that the family

$$\mathcal{D} := \{(R^{[m]}, R^{[m]}) \mid m \in \mathbb{N}\}$$

is not of finite cohomology either. Figure out the difference of the class $\mathcal{D}$ and the classes presented in Example 1.9.

C) Let $R = K[X, Y]$ be a polynomial ring in two indeterminates over a field K furnished with its standard grading. For each positive integer $m \in \mathbb{N}$ let $M^{[m]} := R(-m) \oplus R(m)$. Calculate the Hilbert polynomials $P_{M^{[m]}}$ for all $m \in \mathbb{N}$ and show that the class

$$\mathcal{E} := \{(R, M^{[m]}) \mid m \in \mathbb{N}\}$$

is not of finite cohomology. Compare the situation with the one described in Example 1.9.

2. Supporting Degrees of Cohomology

Let $d \in \mathbb{N}$ and let $(R, M) \in \mathcal{M}^d$. For each $i \in \mathbb{N}_0$ we now aim to look at the *supporting degrees* of the local cohomology modules $H^i_{R_+}(M)$ and the R_+-transform modules $D^i_{R_+}(M)$ of M with respect to R_+, hence at the integers $n \in \mathbb{Z}$ for which $h^i_M(n)$ respectively $d^i_M(n)$ does not vanish. Similarly, if $t \in \mathbb{N}_0$ and $(X, \mathcal{F}) \in \mathcal{S}^t$ we aim to focus at the integers n for which $h^i_{\mathcal{F}}(n)$ does not vanish. We start with a few basic notions concerning graded rings and modules.

2.1. **Notation and Reminder.** A) *(Generating Degrees of Graded Modules)* Let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra, where K is a field. Keep in mind that $R = K[x_1, x_2, ..., x_r]$ for finitely many homogeneous elements $x_1, x_2, ..., x_r \in R_1$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be an arbitrary graded R-module. Keep in mind that $\dim_K(M_n) < \infty$ for all $n \in \mathbb{Z}$ and $M_n = 0$ for all $n \ll 0$ provided that M is in addition finitely generated. Let us recall the notion of *generating degree* of M (see [Br-Fu-Ro] (9.6)D)) defined as

$$\operatorname{gendeg}(M) := \inf\{t \in \mathbb{Z} \mid M = \sum_{n \leq t} RM_n\}.$$

Keep in mind the followeing facts:

a) If M is finitely generated, then $\operatorname{gendeg}(M) < \infty$.
b) If $\operatorname{gendeg}(M) \leq n \in \mathbb{Z}$, then $M_{n+k} = R_k M_n$ for all $k \in \mathbb{N}$.

B) *(Beginnings and Ends of Graded Modules)* Keep the notations and hypotheses of part A) and let us introduce the *beginning* and the *end* of M, which are defined respectively by

$$\operatorname{beg}(M) := \inf\{n \in \mathbb{Z} \mid M_n \neq 0\},$$
$$\operatorname{end}(M) := \sup\{n \in \mathbb{Z} \mid M_n \neq 0\}.$$

Observe the following facts:

a) If $M \neq 0$ is a graded R-module, then $\operatorname{beg}(M) \leq \operatorname{gendeg}(M) \leq \operatorname{end}(M)$.
b) If $M \neq 0$ is a finitely generated graded R-module, then

$$-\infty < \operatorname{beg}(M) \leq \operatorname{gendeg}(M) < \infty.$$

Now, we remind the notion of *Castelnuovo-Regularity* which shall play a fundamental rôle in these lectures.

2.2. **Reminder and Definition.** A) *(Castelnuovo-Mumford Regularity of Graded Modules)* Let $d \in \mathbb{N}$, let $(R, M) \in \mathcal{M}^d$ and let $l \in \mathbb{N}_0$. We define the *(Castelnuovo-Mumford) regularity of M at and above level l* by

$$\operatorname{reg}(M) := \sup\{\operatorname{end}(H^i_{R_+}(M)) + i \mid i \geq l\}.$$

The *(Castelnuovo-Mumford) regularity of M (at all)* is defined as

$$\operatorname{reg}(M) := \operatorname{reg}^0(M).$$

Now, on use of the statements a)-d) of 1.4 A) one verifies immediately that

a) $\mathrm{reg}^l(M) < \infty$.
b) $\mathrm{reg}^l(M) = -\infty$ if and only if $l > d$.
c) $\mathrm{reg}^l(M(n)) = \mathrm{reg}^l(M) - n$ for all $n \in \mathbb{Z}$.
d) If $k \in \{0, 1, \ldots, l\}$, then $\mathrm{reg}^l(M) \leq \mathrm{reg}^k(M)$.

Let us recall the following most important fact, whose proof may be found in [Br8] (Proposition 3.4) or in [Br-Sh1] (Theorem 15.3.1) for example.

e) $\mathrm{gendeg}(M) \leq \mathrm{reg}(M)$.

Finally, let us mention the following facts on the behaviour of regularity in short exact sequences, which follow easily from the corresponding observations made in (1.4)C). So, let

$$\mathbb{S} : 0 \to L \to M \to N \to 0$$

be an exact sequence of finitely generated graded R-modules. Then

f) $\mathrm{reg}(L) \leq \max\{\mathrm{reg}(M), \mathrm{reg}(N) + 1\}$.
g) $\mathrm{reg}^{l+1}(L) \leq \max\{\mathrm{reg}^{l+1}(M), \mathrm{reg}^l(N) + 1\}$.
h) $\mathrm{reg}^l(M) \leq \max\{\mathrm{reg}^l(L), \mathrm{reg}^l(N)\}$.
i) $\mathrm{reg}^l(N) \leq \max\{\mathrm{reg}^{l+1}(L) - 1, \mathrm{reg}^l(M)\}$.

B) *(Castelnuovo-Mumford Regularity of Sheaves)* Let $t \in \mathbb{N}_0$, let $(X, \mathcal{F}) \in \mathcal{S}^t$ and let $k \in \mathbb{N}_0$. We define the *(Castelnuovo-Mumford) regularity of $\mathcal{F}$ above level k* by

$$\mathrm{reg}_k(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid H^i(X, \mathcal{F}(r - i)) = 0, \quad \forall i > k\}.$$

The *(Castelnuovo-Mumford) regularity of $\mathcal{F}$ (at all)* is defined as

$$\mathrm{reg}(\mathcal{F}) := \mathrm{reg}_0(\mathcal{F}).$$

Now, on use of statements a), b) and c) of 1.5 C) it follows at once that

a) $\mathrm{reg}_k(\mathcal{F}) < \infty$.
b) $\mathrm{reg}_k(\mathcal{F}) = -\infty$ if and only if $k \geq t$.
c) $\mathrm{reg}_k(\mathcal{F}(n)) = \mathrm{reg}_k(\mathcal{F}) - n$ for all $n \in \mathbb{Z}$.
d) If $m \in \{0, 1, \ldots, k\}$ then $\mathrm{reg}_k(\mathcal{F}) \leq \mathrm{reg}_m(\mathcal{F})$.

Moreover, the concepts of regularity for graded modules and sheaves may be easily related on use of the observation made in 1.4 B)f) and the consequence of the Serre-Grothendieck Correspondence observed in 1.5 C).

e) If $(R, M) \in \mathcal{M}^{t+1}$ with $X = \mathrm{Proj}(R)$ and $\mathcal{F} = \widetilde{M}$, then

$$\mathrm{reg}_k(\mathcal{F}) = \mathrm{reg}^{k+2}(M).$$

C) *(Regularity and Global Generation of Sheaves)* On may wonder, whether statement e) of part A) has some analogue in the sheaf theoretic context. This is indeed true, and we briefly recall the corresponding facts. For readers who want to get a detailed and selfcontained approach to the subject, we recommend to consult [Br8] (3.8)-(3.13). Let the notations and hypotheses be as in Part B). We say, that the sheaf $\mathcal{F}$ is *generated by global sections* if for each point $x \in X$ the stalk $\mathcal{F}_x$ of the sheaf $\mathcal{F}$ at x is generated over the local ring $\mathcal{O}_{X,x}$ by germs of global sections $\gamma \in \Gamma(X, \mathcal{F})$. So if

$$\bullet_x : \Gamma(X, \mathcal{F}) \to \mathcal{F}_x, \quad \gamma \mapsto \gamma_x, \quad \forall \gamma \in \Gamma(X, \mathcal{F})$$

denotes the map of taking germs, the sheaf $\mathcal{F}$ is generated by its global sections if and only if for each $x \in X$ there is some set $\mathcal{T} \subseteq \Gamma(X, \mathcal{F})$ such that

$$\mathcal{F}_x = \sum_{\gamma \in \mathcal{T}} \mathcal{O}_{X,x} \gamma_x.$$

Then the announced analogue of statement A)e) says:

a) For each $n \geq \operatorname{reg}(\mathcal{F})$ the n-fold twist $\mathcal{F}(n)$ of $\mathcal{F}$ is generated by its global sections.

We now define the notion of *Cohomological Pattern* of a coherent sheaf, which is directly related to the vanishing degrees of cohomology.

2.3. **Definition.** *(Cohomological Patterns)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. We define the *cohomological pattern of the pair* $(X, \mathcal{F})$ (or simply) *of the sheaf* $\mathcal{F}$ as the set

$$\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F}) := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid H^i(X, \mathcal{F}(n)) \neq 0\}$$

of all pairs $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ such that the cohomology table $h_{\mathcal{F}}$ of $\mathcal{F}$ has a non-zero entry at (i, n).

We now formulate the following *Structure Theorem for Cohomological Patterns*, whose proof os given in [Br-He].

2.4. **Theorem.** *Let $t \in \mathbb{N}_0$. Then, a set $\mathcal{P} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is the cohomological pattern of a pair $(X, \mathcal{F}) \in \mathcal{S}^t$ if and only if the following six requirements are satisfied;*

(i) $\sup\{i \in \mathbb{N}_0 \mid \exists n \in \mathbb{Z} : (i, n) \in \mathcal{P}\} = t$;
(ii) $\exists n \in \mathbb{Z} : (0, n) \in \mathcal{P}$;
(iii) $\forall (i, n) \in \mathcal{P} : \exists k \geq i : (k, n - k + i - 1) \in \mathcal{P}$;
(iv) $\forall (i, n) \in \mathcal{P} : \exists l \leq i : (k, n - l + i + 1) \in \mathcal{P}$;
(v) $\forall i \in \mathbb{N}, \quad \forall n \gg 0 : (i, n) \notin \mathcal{P}$;
(vi) $\forall i \in \mathbb{N}_0 : \#\{n \in \mathbb{Z} \mid (i, n) \in \mathcal{P}, (i, n - 1) \notin \mathcal{P}\} < \infty$.

2.5. **Remark.** A) *(Around Cohomological Patterns)* Let the notations be as in (2.3) and (2.4). One might present the cohomological pattern $\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F})$

of the sheaf of the pair $(X, \mathcal{F}) \in \mathcal{S}^t$ (resp. of the sheaf of $\mathcal{O}_X$-modules $\mathcal{F} \neq 0$ in a diagramm with horizontal n-axis and vertical i-axis, marking the place $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ by $\bullet$ if $(i, n) \in \mathcal{P}$ and by $\circ$ otherwise. Then, the five statements (i)-(v) of Theorem (2.16) respectively say:

a) One finds a $\bullet$ on the row at level t and no $\bullet$ on a row with level strictly higher than t.
b) One finds a $\bullet$ on the bottom row.
c) If there is a diagonal consisting entirely of $\circ$'s above a certain level i, there are no $\bullet$'s right of this diagonal above level i.
d) If there is a diagonal consisting entirely of $\circ$'s below a certain level i, there are no $\bullet$'s left of this diagonal below level i.
e) Exept on the bottom row one finds only $\circ$'s far out to the right.

Observe in particular, that as a consequence of these properties of $\mathcal{P}$ we get:

e) If there is a $\bullet$ on the bottom level, then right of it on the bottom level there are only $\bullet$'s.
f) If there is a $\bullet$ on the top level t, then left of this $\bullet$ there are only $\bullet$'s at level t.

$$
\begin{array}{ccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\
\circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

B) *(Cohomological Tameness)* For the moment, let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be an arbitrary Noetherian homogeneous ring (so that R_0 is Noetherian and R is generated over R_0 by finitely many elemts of degree 1). Let M be a finitely generated graded R-module and let $\mathcal{F} = \widetilde{M}$ be the coherentb sheaf of $\mathcal{O}_X$-modules induced by M. Again, for each $n \in \mathbb{Z}$ let $\mathcal{F}(n) = \widetilde{M(n)}$ denote the n-th twist of $\mathcal{F}$. We say that the sheaf $\mathcal{F}$ is *cohomologically tame at level* i, if one of the following requirements is satisfied:

a) $H^i(X, \mathcal{F}(n)) \neq 0$ for all $n \ll 0$;
b) $H^i(X, \mathcal{F}(n) = 0$ for all $n \ll 0$.

By the Serre-Grothendieck Correspondence this is equicalent to the fact, that one of the following two requirements is satisfied:

c) $H^{i+1}_{R_+}(M))_n \neq 0$ for all $n \ll 0$,

d) $H^{i+1}_{R_+}(M))_n = 0$ for all $n \ll 0$,

where $H^{i+1}_{R_+}(M)_n$ denotes the n-th graded component of the $(i+1)$-st local cohomology module of M with respect to the irrelevant ideal $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$ of R. We also say in this situation, that the graded R-module M is *cohomologically tame* at level $i+1$. We say that the coherent sheaf $\mathcal{F}$ is *cohomologically tame* at all, if it is tame at all levels $i \in \mathbb{N}_0$. Correspondingly we say that the finitely generated graded R-module M is *cohomologically tame* at all, if is tame at all levels $j \in \mathbb{N}$. Now, statement (vi) of Theorem (2.4) says

e) If $(X, \mathcal{F}) \in \mathcal{S}^t$ (so that $R_0 = K$ is a field), then $\mathcal{F}$ is cohomologically tame. In particular, at each level i there are either only finitily many $\bullet$'s or finitely many $\circ$' with negative n-coordinate.

C)*(The Tameness Problem)* For a while it was indeed an open problem, whether all coherent sheaves $\mathcal{F} = \widetilde{M}$ over a projective scheme $X = \mathrm{Proj}(R)$ defined by an arbitrary Noetherian homogeneous ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ are are cohomologically tame or - equivalently - whether all finitely generated graded modules M over a Noetherian homogeneous ring R are cohomologically tame (see [Br4], [Br6]) (at all levels). There are indeed many results, proving tameness of a finitely generated graded module M over a Noetherian homogeneous ring R at particular levels or under certain assumptions on R - or else on M (see [Br6], [Br7], [Br-Fu-Lim], or also [Br-He], [Lim3], [Rot-Seg] for example). Nevertheless in [Ch-Cu-Her-Sr] a striking counter-example is constructed. Namely, it is shown there:

a) There exists a Noetherian homogeneous domain $R = \bigoplus_{n\in\mathbb{N}_0} R_n$, of finite type over the complex field $\mathbb{C}$ with $\dim(R) = 4$ and $\dim(R_0) = 3$ such that $M = R$ is not cohomologically tame at level 2 (or equivalently $\mathcal{O}_{\mathrm{Proj}(R)}$ is not cohomologically tame at level 1).

This immediately shows, that even over polynomial rings over $\mathbb{C}$ the mentioned Tameness Problem finds a negative answer.

D)*(The Realization Problem for Smooth Complex Projective Varieties)* Let $X = \mathrm{Proj}(R)$ be a smooth connected complex projective variety of dimension at least 2, so that R is a Noetherian homogeneous integral $\mathbb{C}$-algebra such that the local ring $\mathcal{O}_{X,x} = R_{(\mathfrak{p})}$ is regular for all $x = \mathfrak{p} \in X = \mathrm{Proj}(R)$. Then, by the Vanishing Theorem of Kodaira [Ko] one has $H^i(X, \mathcal{O}_X(n)) = 0$ for all $i < \dim(X) = \dim(R) - 1$ and all $n < 0$. By another result of Mumford and Ramanujam [Mu2] one has the same vanishing statement for $i = 1$ under the weaker assumption that X is normal. So, one is naturally lead to ask the following realization question:

a) Let $t \geq 2$ be an integer and let $\mathcal{P} \subseteq \{0, 1, \cdots t\} \times \mathbb{Z}$ be a set which satisfies the pattern requirements (2.4)(i)-(vi) and the additional *positivity* condition that $(i, n) \notin \mathcal{P}$ if $i < t$ and $n < 0$. Does there exist a smooth (or only normal) complex projective variety X (of dimension t) such that $\mathcal{P}_X(X, \mathcal{O}_X) = \mathcal{P}$?

We do not know the answer to this question, even in the surface case, that is in the case $t = 2$. In [M] a method is given, which allows to realize by smooth surfaces a great variety of positive patters as discussed above. We also should mention that by the *Non-Rigidity Theorem* of Evans-Griffiths [Ev-Gri] (see also [Mi-N-P]) there are realization results of the above type in which indeed more than the cohomological pattern is described. Nevertheless, these results allow a realization only up to an eventual shift do not allow to control the last supporting degree the top cohomology groups. Therefore they do not answer our question. Another, local realization result, similar to those just quoted, is given in [Br-Sh2].

We now return to cohomological patterns of pairs $(X, \mathcal{F}) \in \mathcal{S}^t$, that is we concentrate again to the case where $X = \text{Proj}(R)$ is a projective scheme over some field K and hence induced by some Noetherian homogeneous K-algebra. A natural (and fundamental) question is to ask for the lowest level i at which the cohomological pattern $\mathcal{P}_{\mathcal{F}} = \mathcal{P}(X, \mathcal{F})$ of $\mathcal{F}$ has infinitely many entries (i, n) with $n < 0$. By the tameness-property (vi) of Theorem (2.4) it is equivalent to ask for the lowest level $i \leq t$ such that there are only finitely many negative integers $n < 0$ with $(i, n) \notin \mathcal{P}_{\mathcal{F}}$.

2.6. **Definition and Remark.** A) *(Cohomological Finiteness Dimension of a Coherent Sheaf)* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Then, the *(cohomological) finiteness dimension of (X with respect to)* $\mathcal{F}$ is defined as

$$\begin{aligned} \text{fdim}(\mathcal{F}) &:= \inf\{i \in \mathbb{N}_0 \mid \#\{n < 0 \mid h^i_{\mathcal{F}}(n) \neq 0\} = \infty\} \\ &= \inf\{i \in \mathbb{N}_0 \mid \#\{n < 0 \mid h^i_{\mathcal{F}}(n) = 0\} < \infty\}. \end{aligned}$$

So $\text{fdim}(\mathcal{F})$ is the lowest level on which there are infinitely many $\bullet$'s at places with negative n-coordinate or - equivalently - only finitely many $\circ$'s with negative n-coordinate.

B) *(Algebraic Characterization Cohomological Finiteness Dimension)* Keep the notations and hypotheses of part A) and let $(R, M) \in \mathcal{M}^{t+1}$ such that $(X, \mathcal{F}) = (\text{Proj}(R), \widetilde{M})$. In Chapter 9 of [Br-Sh1] the R_+*-finiteness dimension of* M is introduced as the invariant

a) $f_{R_+}(M) := \inf\{j \in \mathbb{N} \mid H^j_{R_+}(M) \text{ not finitely generated}\}$.

As the K-vector spaces $H^j_{R_+}(M)_n$ are finitely generated and vanish for all $n \gg 0$, it follows, that

b) $f_{R_+}(M) = \inf\{j \in \mathbb{N} \mid \#\{n < 0 \mid h^j_M(n) \neq 0\} = \infty\}$.

Now on use of the relations 1.5 D)a),b) it follows that

c) $\mathrm{fdim}(\mathcal{F}(M)) = f_{R_+}(M) - 1$.

The finiteness dimension of a coherent sheaf $\mathcal{F}$ of modules over a projective scheme X over some field K is by its very definition a "global invariant" as it is defined by means of the vanishing and non-vanishing of the cohomology groups $H^i(X, \mathcal{F}(n))$ which at their turn are global invariants of the twisted sheaf $\mathcal{F}(n)$. But nevertheless these global invariants are deeply related to a local invariant of the sheaf $\mathcal{F}$, which we shall define now.

2.7. **Definition and Remark.** A) *(Subdepth of a Coherent Sheaf).* Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Let $(R, M) \in \mathcal{M}^{t+1}$ with $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. We consider the *set of closed points of* $X = \mathrm{Proj}(R)$, that is the set

$$\mathrm{m}X = \mathrm{mProj}(R) := \{x = \mathfrak{p} \in X = \mathrm{Proj}(R) \mid \dim(R/\mathfrak{p}) = 1\}$$

and define the *subdepth of (X with respect to)* $\mathcal{F}$ by

$$\delta(\mathcal{F}) := \inf\{\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in \mathrm{m}X\}.$$

B) *(Algebraic Description of Subdepth)* Keep the above notations and hypotheses. If $x = \mathfrak{p} \in X = \mathrm{Proj}(R)$ we have

$$\mathcal{O}_{X,x} = R_{(\mathfrak{p})}, \quad \mathcal{F}_x = M_{(\mathfrak{p})}$$

where $\bullet_{(\mathfrak{p})}$ denotes homogeneous localization at $\mathfrak{p}$. Therfore, we also may write

$$\delta(\mathcal{F}) = \inf\{\mathrm{depth}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) \mid \mathfrak{p} \in \mathrm{mProj}(R)\}.$$

Now, the two previously introduced invariants are related be the *Vanishing Theorem of Severi-Enriques-Zariski-Serre*:

2.8. **Theorem.** *Let* $t \in \mathbb{N}_0$ *and let* $(X, \mathcal{F}) \in \mathcal{S}^t$. *Then*

$$\mathrm{fdim}(\mathcal{F}) = \delta(\mathcal{F}).$$

2.9. **Remark.** A) *(On the Proof of the Vanishing Theorem of Severi-Enriques-Zariski-Serre).* One approach to prove Theorem 2.8 is to use Serre-Duality (see [H1] for example). This approach corresponds essentially to Serre's original proof in [Se]. Another approach (which leads indeed even to a quantitive version of the requested result) for projective schemes over algebraically closed fields is found in [Br-Fu-Ro] (see Chapters 10 for an algebraic version and Chapter 12 for the translation to sheaf theory). It is easy, to drop the hypothesis of algebraically closed ground field (see [Br8] (7.11) D)). Another approach is to use a much more general algebraic result: the *Graded Version of Grothendieck's Finiteness Theorem* or - in fact even more general - the *Graded Version of Falting's Annihilator Theorem* (for both see [Br-Sc1] (13.1.17)). Under the hypotheses and in the notations of (2.6) and (2.7), one has only to show that

a) $\delta(\mathcal{F}) + 1 = \inf\{\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{ht}\big((R_+ + \mathfrak{p})/\mathfrak{p}\big) \mid \mathfrak{p} \in \mathrm{Proj}(R)\}$, Indeed from this the mentioned graded version of Grothendieck's finiteness theorem allows to deducethe relation $f_{R_+}(M) = \delta(\mathcal{F})$, and by (2.6) B)c) one obtains Theorem 2.8. But statement a) follows easily from the well known fact that

b) $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{depth}_{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}), \quad \forall \mathfrak{p} \in \mathrm{Proj}(R)$

(whose proof might be a recommendable ecerxise) and bearing in mind the last observation made in 2.6 B).

B) *(Around the History of the Vanishing Theorem of Severi-Enriques-Zariski-Serre).* We briefly aim to explain the string of names associate to this theorem. There are three preceeding results to the mentioned theorem shown by Severi [Sev] 1942, Enriques [En] 1949 and Zariski [Z] 1952 - thus all three of them in the "pre-cohomological aera" of Algebraic Geometry - and formulated in the language of linear systems and divisors. If one translates these results to our cohomological language, they respectively correspond to the following special cases of Theorem 2.8

a) (Severi 1942) $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ is a smooth surface in complex projective 3-space and $\mathcal{F} := \omega_X$ is the canonical bundle of X
b) (Enriques 1949) $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ is a smooth hyperplane in complex projective r-space and $\mathcal{F} := \omega_X$ is the canonical bundle of X.
c) (Zariski 1952) X is a normal projective variety over an algebraically closed field and $\mathcal{F} = \mathcal{L}$ is an ample line bundle on X.

In his seminal work [Se] in 1955, Serre proved Theorem 2.8 for arbitrary projective varieties over algebraically closed fields and arbitrary coherent (algebraic) sheaves over such varieties. In that same paper he actually introduced sheaf theory and sheaf cohomology for arbitrary algebraic varieties over algebraically closed fields. This finally paved the way to Grothendiek's scheme theoretic and functorial approach to Algebraic Geometry [Gro-D]. As we mentioned already in part A), Theorem 2.8 found a further (algebraic) generalization: Grothendieck's Finiteness Theorem for Local Cohomology. A next step of generalization is Falting's Annihilator Theorem for Local Cohomology (see [Fa1] or [Br-Sh1] Chaper 9), whose graded version we already mentioned in part A).

C) *(Cohomological Characterization of Algebraic Vector Bundles)* One of the basic applications of the Vanishing Theorem of Severi-Enriques-Zariski-Serre is a cohomological characterization of algebraic vector bundles. To recall this application, we assume that X is a smooth (irreducible) projective variety of dimension $t > 0$ over an algebraically closed field, so that $X = \mathrm{Proj}(R)$, where $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ is an Noetherian homogeneous integral domain with the property that $R_{(\mathfrak{p})} = \mathcal{O}_{X,x}$ is a regular local ring of dimension t for all $x = \mathfrak{p} \in \mathrm{m}X = \mathrm{mProj}(R)$. Now, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, by the *Formula of Auslander-Buchsbaum* it follows easily, that for every $x \in \mathrm{m}X$ the finitely generated $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is free if and only if $\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = t$. The coherent sheaf $\mathcal{F}$ is an *algebraic vector bundle over* X if and conly if the stalk $\mathcal{F}_x$ of $\mathcal{F}$ is a free module over the local ring $\mathcal{O}_{X,x}$ for all $x \in \mathrm{m}X$. So, by Theorem 2.8 we can say

If X is a smooth projective variety over an algebraically closed field K, a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$- modules is an algebraic vector bundle if and only if the cohomological finiteness dimension $\mathrm{fdim}(\mathcal{F})$ *of $\mathcal{F}$ takes the (maximally possible) value* $\dim(X)$.

A more detailed presentation of the relation between sheaf cohomology and algebraic vector bundles may be found in [Br8] (7.11)-(7.13).

3. Modules of Deficiency

In this section we introduce an important tool for the treatment of local cohomology modules, the so called *Modules of Deficiency* (or just *Deficiency Modules* for short). More precicely, to each each finitely generated graded module M over a Noetherian homogeneous algebra R over a field K we introduce a family if finitely generated graded R-modules $\big(K^i(M)\big)_{i\in\mathbb{N}_0}$, such that $K^i(M)$ is K-dual to the corresponding local cohomology module $H^i_{R_+}(M)$ for each $i \in \mathbb{N}_0$. In our *Main Theorem on Modules of Deficiency* we collect all the relevant properties of deficiency modules. As an application, we shall be able to introduce the concept of *i-th Cohomological Hilbert Polynomial p^i_M of M* and the notion of *i-th Cohomological Postulation Number ν^i_M of M* of a finitely generated graded module over a Noetherian homogeneous K-algebra R. A basic issue is the fact that the Castelnuovo-Mumford regularity of the i-th deficiency module gives a lower bound for the i-th cohomological postulation number. We also shall consider the most important class of deficiency modules: the *Canonical Modules* $K(M) := K^{\dim_R(M)}$. We shall establish one property of these modules, which has to be used to prove the main result of Section 4. A more detailed and complete treatment of all these and further related result may be found in [Br8].

3.1. **Construction and Exercise.** A) *(Graded Dual Modules)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \dots$ be a Noetherian homogeneous K-algebra and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. We consider the K-dual of M, that is the K-vector space

$$M^\vee := \mathrm{Hom}_K(M, K)$$

of all K-linear maps $h : M \to K$. By means of the scalar multiplication defined by

$$xh := h \circ x\mathrm{Id}_M, \quad \forall x \in R, \quad \forall h \in M^\vee,$$

the K-vector space $M^\vee$ is turned into an R-module. We consider the subset

$$D(M) := \{h \in M^\vee \mid \#\{n \in \mathbb{Z} \mid h(M_n) \neq 0\} < \infty\} \subseteq M^\vee$$

consisting of all K-linear maps $h : M \to K$ which vanish on almost all graded components of M. Moreover, for each $t \in \mathbb{Z}$ we define the subset

$$D(M)_t := \{h \in M^\vee \mid h(M_n) = 0, \quad \forall n \neq -t\}.$$

Prove the following statements:

a) $D(M) \subseteq M^\vee$ is an R-submodule.
b) For all $t \in \mathbb{Z}$ the set $D(M)_t \subseteq D(M)$ is a K-subspace.
c) The family $(D(M)_t)_{t\in\mathbb{Z}}$ of K-subspaces $D(M)_t \subseteq D(M)$ defines a grading on the R-module $D(M)$.
d) For all $t \in \mathbb{Z}$ there is an isomorphism of K-vector spaces

$$\tau^M_t : (M_{-t})^\vee := \mathrm{Hom}_K(M_{-t}, K) \xrightarrow{\cong} D(M)_t$$

given by

$$\tau_t^M(h)(m) := h(m_{-t}), \quad \forall h \in (M_{-t})^\vee, \quad \forall m := (m_n)_{n\in\mathbb{Z}} \in M = \bigoplus_{n\in\mathbb{Z}} M_n.$$

e) For all $r, t \in \mathbb{Z}$ we have $D(M(r))_t = D(M)_{t-r}$.

From now on, we always furnish the R-module $D(M)$ with the grading mentioned in statement c), hence write

$$D(M) = \bigoplus_{t\in\mathbb{Z}} D(M)_t,$$

and call $D(M)$ the *graded (K-) dual of M*. Observe that by statement e) we have

f) $D(M(r)) = D(M)(-r), \quad \forall r \in \mathbb{Z}$.

B) *(The Graded Duality Functor)* Keep the notations and hypotheses of part A) and let $h : M \to N$ be a homomorphism of graded R-modules. Show that there is a homomorphism of graded R-modules

$$D(h) : D(N) \to D(M), \quad f \mapsto f \circ h, \quad \forall f \in D(N).$$

This homomorphisms of graded R-modules is called the *graded (K-) dual of h*. Prove that we have a *contravariant, R-linear, exact functor of graded R-modules*

$$D(\bullet) : (M \xrightarrow{h} N) \leadsto \big(D(N) \xrightarrow{D(h)} D(M)\big),$$

the *functor of taking graded (K-)duals* or the *graded duality functor (with respect to K)*.

C) *(First Properties of Graded Duality Functors)* Keep the notations and hypotheses of parts A) and B). Show the following

a) For all $t \in \mathbb{Z}$ there is a natural equivalence of contravariant functors from graded R-modules to K-vector spaces

$$\tau_t^M : (\bullet_{-t})^\vee \xrightarrow{\cong} D(\bullet)_t : M \leadsto \big((M_{-t})^\vee \xrightarrow{\tau_t^M} D(M)_t\big),$$

where τ_t^M is defined as in statement A)d).

b) There is a natural transformation of covariant functors of graded R-modules

$$\gamma : \bullet \to D(D(\bullet)) : M \leadsto \big(M \xrightarrow{\gamma^M} D(D(M))\big),$$

where the homomorphism $\gamma^M : M \to D(D(M))$ is given by

$$\gamma^M(m)(f) = f(m), \quad \forall m \in M, \quad \forall f \in D(M).$$

D) *(Base Ring Independence of Graded Duals)* Keep the notations of part A) and assume that $\mathfrak{a} \subsetneq R$ is a proper graded ideal such that $\mathfrak{a}M = 0$. Show that the graded R-module $D(M)$ satisfies $\mathfrak{a}D(M) = 0$ and is independent on whether we consider M as an R-module or an $R/\mathfrak{a}$-module.

We now establish a few basic facts about graded duals over graded K-algebras.

3.2. **Exercise and Definition.** A) *(Modules with Finite Components)* Let the notations and hypotheses be as in 3.1. We say that a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ has *finite components* if

$$\dim_K(M_n) < \infty, \quad \forall n \in \mathbb{Z}.$$

We denote the class of graded R-modules with finite components by $\mathbb{F}_R$. Use the well known properties of taking duals of finite-dimensional vector spaces and the natural equivalences of 3.1 C)a) to show the following statements:

a) If $M \in \mathbb{F}_R$, then $\dim_K(D(M)_t) = \dim_K(M_{-t})$ for all $t \in \mathbb{Z}$.
b) If $M \in \mathbb{F}_R$, then $D(M) \in \mathbb{F}_R$.
c) If $M \in \mathbb{F}_R$, the canonical map $\gamma^M : M \to D(D(M))$ (see (3.1)C)b)) is an isomorphism of graded R-modules.

B) *(Equihomogeneous Ideals)* Keep the above notations and hypotheses. An ideal $\mathfrak{a} \subseteq R$ is said to be *equihomogeneous* if it is generated by homogeneous elements of the same degree. We now are intersted in finitely generated equihomogeneous ideals. So, let $s \in \mathbb{Z}$, let $r \in \mathbb{N}$, let $x_1, x_2, \ldots, x_r \in R_s$, let M be a graded R-module and consider the multiplication maps given by these elements, that is the homomorphisms of graded R-modules

$$x_i = x_i\mathrm{Id}_M : M \to M(s), \quad m \mapsto x_i m, (i = 1, 2, \ldots, r).$$

Use the properties of kernels and cokernels of K-linear maps with respect to taking K-duals to show the following facts:

a) $(0 :_M \langle x_1, x_2, \ldots, x_r\rangle)_{-t} = \bigcap_{i=1}^r \mathrm{Ker}(x_i\restriction_{M_{-t}})$ for all $t \in \mathbb{Z}$.
b) There is an isomorphism of graded R-modules

$$D(M)/\langle x_1, x_2, \ldots, x_r\rangle D(M) \xrightarrow{\cong} D(0 :_M \langle x_1, x_2, \ldots, x_r\rangle)$$

defined by

$$u + \langle x_1, x_2, \ldots, x_r\rangle D(M) \mapsto u\restriction_{(0:_M\langle x_1,x_2,\ldots,x_r\rangle)}, \quad \forall u \in D(M).$$

Now, we shall introduce the notions of *Deficiency Functors and Deficiency Modules.*

3.3. **Exercise and Definition.** A) *(Deficiency Functors and -Modules)* Let K be a field and let $R = K\oplus R_1\oplus R_2\ldots$ be a Noetherian homogeneous K-algebra. For each $i \in \mathbb{N}_0$ we define the *i-th deficiency functor* $K^i = K^i(\bullet)$ *(over R)* as the contravariant linear functor of graded R-modules obtained by composing the (graded) local cohomology functor $H^i_{R_+}(\bullet)$ with the graded duality functor $D = D(\bullet)$, thus the functor of graded R-modules given by the assignement

$$(M \xrightarrow{h} N) \rightsquigarrow \big(K^i(M) = D(H^i_{R_+}(N)) \xrightarrow{K^i(h)=D(H^i_{R_+}(h))} D(H^i_{R_+}(M)) = K^i(M)\big).$$

For each graded R-module M, the graded R-module $K^i(M)$ is called the *i-th deficiency module of M.*

B) *(First Properties of Deficiency Functors)* Keep the notations and hypotheses of part A). Let $i \in \mathbb{N}_0$. Prove the following facts:

a) *(Duals af Deficiency Modules)* There is a natural transformation of covariant functors of graded R-modules

$$\kappa^i : H^i_{R+}(\bullet) \to D(K^i(\bullet)) : M \rightsquigarrow \big(H^i_{R_+}(M) \xrightarrow{\kappa^{i,M}:=\gamma^{H^i_{R_+}(M)}} D(K^i(M))\big),$$

where the homomorphism

$$\gamma^{H^i_{R_+}(M)} : H^i_{R_+}(M) \to D(D(H^i_{R_+}(M))) = D(K^i(M))$$

is defined according to (3.1)C)b).

b) *(Base Ring Independence of Deficiency Modules)* If M is a graded R-module and $\mathfrak{a} \subsetneq R$ is a proper graded ideal with $\mathfrak{a}M = 0$ we have $\mathfrak{a}K^i(M) = 0$. In addition (up to isomorphism of graded R-modules) the module $K^i(M)$ remains the same if we consider M as as a graded $R/\mathfrak{a}$-module.

C) *(Deficiency Modules of Finitely Generated Modules)* Let the notations be as in parts A) and B) and assume that the graded R-module M is finitely generated. Prove the following facts:

a) $H^i_{R_+}(M)$ and $K^i(M)$ belong to the class $\mathbb{F}_R$ (see (3.2)A)).
b) $\dim_K(K^i(M))_n = h^i_M(-n)$ for all $n \in \mathbb{Z}$.
c) $\mathrm{beg}(K^i(M)) = -\mathrm{end}(H^i_{R_+}(M)) > -\infty$.
d) $\sup\{i \in \mathbb{N}_0 \mid K^i(M) \neq 0\} = \dim_R(M)$.
e) The natural homomorphism of graded R-modules of (3.3)B)a) becomes an isomorphism

$$\kappa^{i,M} : H^i_{R_+}(M) \xrightarrow{\cong} D(K^i(M)).$$

D) *(The Deficiency Sequence)* Keep the obove notations and hypothesis and let

$$\mathbb{S} : 0 \to N \xrightarrow{h} M \xrightarrow{l} P \to 0$$

be an exact sequence of graded R-modules. Apply the *exact graded cohomology sequence with respect to R_+ and associated to* $\mathbb{S}$ (see [Br-Fu-Ro](8.26)A) for example) and then apply the graded duality functor $D(\bullet)$ to the resulting sequence. Show that you end up with a natural exact sequence of graded

R-modules

$$
\begin{array}{l}
\cdots \longrightarrow K^{i+1}(M) \xrightarrow{K^{i+1}(h)} K^{i+1}(N) \xrightarrow{\varepsilon^{i}_{\mathbb{S}}} \\
K^{i}(P) \xrightarrow{K^{i}(l)} K^{i}(M) \xrightarrow{K^{i}(h)} K^{i}(N) \xrightarrow{\varepsilon^{i-1}_{\mathbb{S}}} \\
K^{i-1}(P) \longrightarrow \cdots \\
\cdots \longrightarrow K^{1}(M) \xrightarrow{K^{1}(h)} K^{1}(N) \xrightarrow{\varepsilon^{0}_{\mathbb{S}}} \\
K^{0}(P) \xrightarrow{K^{0}(l)} K^{0}(M) \xrightarrow{K^{0}(h)} K^{0}(N) \longrightarrow 0
\end{array}
$$

in which the maps $\varepsilon^i_{\mathbb{S}}$ are induced by the corresponding connecting homomorphism in the cohomology sequence associated to $\mathbb{S}$. We call this sequence the *deficiency sequence associated to* $\mathbb{S}$.

E) *(Socles of Local Cohomology Modules)* Let R be as above. For any graded R-module U one defines the *socle of* U as the graded submodule

$$\mathrm{soc}(U) := (0 :_U R_+) \subseteq U.$$

Observe that $R_+\mathrm{soc}(U) = 0$, so that $\mathrm{soc}(U)$ is a vector space over $R/R_+ \cong K$ and the R-submodules of $\mathrm{soc}(U)$ are precicely the K-vector subspaces.

Now, let M be a finitely generated graded R-module and chose elements $x_1, x_2, \ldots, x_r \in R_1$ such that

$$\langle x_1, x_2, \ldots, x_r \rangle = R_+.$$

Let $i \in \mathbb{N}_0$ and use the ideas of (3.2)B) to prove the following statements:

a) There is an isomorphism of graded R-modules

$$K^i(M)/R_+K^i(M) \xrightarrow{\cong} D\big(\mathrm{soc}(H^i_{R_+}(M))\big).$$

b) $K^i(M)$ is finitely generated if and only if we have $\mathrm{end}(H^i_{R_+}(M)) < \infty$ and $\mathrm{soc}(H^i_{R_+}(M))$ is finitely generated.

In the next exercise we prepare some arguments which will be used repeatedly later.

3.4. **Exercise.** A) *(Deficiency Modules and Torsion).* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra and let M

be a graded R-module. Let us recall, that an R-module T is said to be R_+-*torsion*, if $T = \Gamma_{R_+}(T)$. Use the corresponding statements on induced homomorphisms between local cohomology modules (see [Br-Fu-Ro] (3.18) for example) to prove:

a) If M is R_+-torsion, then $K^i(M) = 0$ for all $i \in \mathbb{N}$.
b) If M is finitely generated, then $K^0(M)$ is R_+-torsion, finitely generated and satisfies $\dim_K(K^0(M)) = \dim_K(H^0_{R_+}(M)) < \infty$.
c) If $N \subseteq M$ is a graded submodule which is R_+-torsion and $p : M \to M/N$ is the canonical homomorphism, then the induced homomorphism $K^i(p) : K^i(M/N) \to K^i(M)$ is an isomorphism if $i > 0$ and a monomorphism if $i = 0$.

B) *(Deficiency Modules and Non-Zero Divisors)* Let the notations and hypotheses be as in part A). For any R-module N let

$$\mathrm{NZD}_R(N) := \{x \in R \mid xn \neq 0. \quad \forall n \in n \setminus \{0\}\}$$

denote the set of *non-zero divisors of R with respect to N*. Now, let $t \in \mathbb{N}$ and let $x \in R_t \cap \mathrm{NZD}_R(M)$. If we form the deficiency sequence associated to the short exact sequence of graded R-modules

$$\mathbb{S} : 0 \to M(-t) \xrightarrow{x} M \xrightarrow{p} M/xM \to 0$$

and write $\varepsilon^i_{M,x} := \varepsilon^i_{\mathbb{S}}$ for all $i \in \mathbb{N}_0$ (see (3.3)D), we can say:

a) For each $i \in \mathbb{N}_0$ there is an exact sequence of graded R-modules

$$K^{i+1}(M) \xrightarrow{x} K^{i+1}(M)(t) \xrightarrow{\varepsilon^i_{M,x}} K^i(M/xM) \xrightarrow{K^i(p)} K^i(M) \xrightarrow{x} K^i(M)(t).$$

Consequently

b) For each $i \in \mathbb{N}_0$ there is a short exact sequence of graded R-modules

$$0 \to \big(K^{i+1}(M)/xK^{i+1}(M)\big)(t) \to K^i(M/xM) \to (0 :_{K^i(M)} x) \to 0.$$

Now we are ready to give a first result on the structure of deficiency modules.

3.5. **Proposition.** *Let K be a field, let $R \in \mathbb{N}_0$ and let $R := K[X_1, X_2, \ldots, X_r]$ be a polynomial ring.*

a) *If $i \neq r$, then $K^i(R) = 0$.*
b) *$K^r(M) \cong R(-r)$.*

Proof. As R is CM, we have $H^i_{R_+}(R) = 0$ for all $i \neq r$. So, statement a) follows from (3.3)C)b).

We prove statement b) by induction on r. If $r = 0$, we have $R = K = H^0_{R_+}(R)$. If we apply (3.3)C)b) with $i = 0$ it follows that $K^0(M) = K = R = R(-0)$.

So let $r > 0$. We consider the polynomial ring

$$R' := K[X_1, X_2, \ldots, X_{r-1}].$$

By induction we have $K^{r-1}(R') \cong R'(-r+1)$. Observe that there is an isomorphism of graded R-modules $R' \cong R/X_rR$. So, by the Base Ring Independence of Deficiency Modules (3.3)B)b) we get an isomorphism of graded R-modules

$$K^{r-1}(R/X_rR) \cong (R/X_rR)(-r+1).$$

If we apply the short exact sequence (3.4)B)b) with $i = r-1$, $x = X_r$, $M = R$ and keep in mind that $K^{r-1}(R) = 0$ we therefore get isomorphisms of graded R-modules

$$K^r(R)/X_rK^r(R) \cong K^{r-1}(R/X_rR)(-1) \cong (R/X_rR)(-r).$$

As a consequence

$$K^r(R)/(R_+)K^r(R) \cong R/(X_rR)(-r)/(R_+)(R/X_rR)(-r) \cong$$
$$\cong \big((R/X_rR)/(R_+)(R/X_r)\big)(-r) \cong (R/R_+)(-r).$$

This shows that $K^r(R)/(R_+)K^r(R)$ is generated by a single element of degree r. As $\operatorname{beg}(K^r(M)) = -\operatorname{end}(H^r_{R_+}(R)) > -\infty$ (see (3.3)C)c)), the Graded Nakayama Lemma implies that $K^r(R) = Ra$ for some $a \in K^r(R)_r$. So, there is an epimorphism of graded R-modules

$$R(-r) \xrightarrow{\pi} K^r(R) \to 0, \quad f \mapsto fa.$$

Now, let $x \in R_t \setminus \{0\}$ for some $t \in \mathbb{N}$. Then $\dim_R(R/xR) < r$ shows that the multiplication map $x : H^r_{R_+}(R)(-t) \to H^r_{R_+}(R)$ is surjective. Therefore the multiplication map $x : K^r(R) \to K^r(R)(t)$ is injective. This shows, that $K^r(R)$ is R-torsion-free and hence of dimension r. This proves, that the epimorphism π is indeed an isomorphism. $\square$

Now, we are ready to prove the following *Main Theorem on Deficiency Modules.*

3.6. **Theorem.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then*

a) *$K^i(M)$ is a finitely generated graded R-module.*
b) *$\dim_K(K^i(M)_n)) = h^i_M(-n)$ for all $n \in \mathbb{Z}$.*
c) *$\operatorname{beg}(K^i(M)) = -\operatorname{end}(H^i_{R_+}(M)) > -\infty$.*
d) *$K^i(M) = 0$ for all $i > \dim_R(M)$.*
e) *$\dim_R(K^i(M)) \leq i$ for all $i \leq \dim_R(M)$ with equality if $i = \dim_R(M)$.*

Proof. "a)“: We find a polynomial ring $S = K[X_1, X_2, \ldots, X_r]$ and a proper graded ideal $\mathfrak{a} \subsetneq S$ such that $R = S/\mathfrak{a}$. According to the Base Ring Independence of Deficiency Modules (3.3)B)b) we may consider M as a graded S-module and hence assume that $R = K[X_1, X_2, \ldots, X_r]$. If $M = 0$ we have

$K^i(M) = 0$. So, let $M \neq 0$. We show by induction on the homological dimension $h := \mathrm{hdim}(M)(\in \mathbb{N}_0)$ of M that $K^i(M)$ is finitely generated. If $h = 0$ we have an isomorphism of graded R-modules

$$M \cong \bigoplus_{k=1}^{s} R(-a_k), \quad a_k \in \mathbb{Z}, \quad \forall k \in \{1, 2, \ldots, s\}, \quad a_1 \leq a_2 \leq \ldots \leq a_s.$$

So, by (3.5) and the additivity of the contravariant functor of graded R-modules $K^i(\bullet)$ we get $K^i(M) = 0$ if $i \neq r$ and $K^r(M) \cong \bigoplus_{k=1}^{s} R(-r + a_k)$.

Now. let $h > 0$ and consider a minimal presentation

$$\mathbb{S} : 0 \to N \to F \to M \to 0, \quad F = \bigoplus_{k=1}^{s} R(-a_k), \quad a_1 \leq a_2 \leq \ldots \leq a_s$$

of M. As $\mathrm{hdim}(N) = \mathrm{hdim}(M) - 1 = h - 1$, by induction $K^j(N)$ is finitely generated for all $j \in \mathbb{N}_0$. By the case $h = 0$ we have $K^j(F) = 0$ for all $j \neq r$ and $K^r(F)$ is a graded free R-module of finite rank. So, the deficiency sequence (3.3)D) associated to $\mathbb{S}$ gives rise to isomorphisms of graded R-modules

$$K^{j+1}(N) \cong K^j(M), \quad \forall j \in \{0, 1, \ldots, r-2\},$$

an epimorphism of graded R-modules

$$K^r(N) \to K^{r-1}(M) \to 0,$$

and a short exact sequence of graded R-modules

$$K^{r+1}(N) \to K^r(M) \to K^r(F).$$

Hence, $K^i(M)$ is finitely generated if $i \leq r$. As $K^i(M) = 0$ if $i > \dim_R(M)$ (see (3.3)C)d)) and as $\dim_R(M) \leq r$, we get our claim.

"b)": This is nothing else than (3.3)C)b).

"c)": This is a restatement of (3.3)C)c).

"d)": This is clear by (3.3)C)d).

"e)": We do not prove this here. Instead we refer to [Br8] (9.7). $\square$

We now use the previous development to introduce *Cohomological Hilbert Polynomials* of graded modules and *Cohomological Serre Polynomials* of coherent sheaves and the related concepts of *Cohomological postulation Numbers.*

3.7. **Remark and Definition.** A) *(Cohomological Hilbert Polynomials)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Fix $i \in \mathbb{N}_0$ and consider the Hilbert polynomial $P_{K^i(M)}$ of the finitely generated graded R-module $K^i(M)$. Then, by the definition of $P_{K^i(M)}$ and by (3.6)b) we have

$$h^i_M(n) = \dim_K(K^i(M)_{-n}) = P_{K^i(M)}(-n), \quad \forall n \ll 0.$$

If we set

$$p^i_M(X) := P_{K^i(M)}(-X)$$

we thus have

$$h^i_M(n) = p^i_M(n), \quad \forall n \ll 0.$$

The polynomial $p^i_M \in \mathbb{Q}[X]$ is called the *i-th cohomological Hilbert polynomial of M*.

B) *(First Properties of Cohomological Hilbert Polynomials)* Let the notations and hypotheses be as in part A). Prove the following facts. (For statement c) see (1.8))

a) $\deg(p^i_M) \leq i-1$ with equality if $i = \dim_K(M) > 0$.

b) $p^i_{M(r)}(X) = p^i_M(r+X)$ for all $r \in \mathbb{Z}$.

c) $P_M(X) = \sum_{i=1}^{\dim_R(M)-1}(-1)^{i-1}p^i_M(X) = \sum_{n\in\mathbb{N}}(-1)^{i-1}p^i_M(X)$.

C) *(Cohomological Postulation Numbers of Graded Modules)* Let the notations and hypotheses be as in parts A) and B). Then clearly

$$\nu^i_M := \inf\{n \in \mathbb{Z} \mid p^i_M(n) \neq h^i_M(n)\} \in \mathbb{Z} \cup \{\infty\}.$$

The number ν^i_M is called the *i-th cohomological postulation number of M*. Prove the following statements:

a) $\nu^i_M = \infty$ if and only if $H^i_{R_+}(M) = 0$.

b) If $\nu^i_M < \infty$, then $\nu^i_M \leq \operatorname{end}(H^i_{R_+}(M))$.

c) $\nu^i_{M(r)} = \nu^i_M - r$ for all $r \in \mathbb{Z}$.

D) *(Cohomological Serre Polynomials)* Let (R, M) be as in parts A) and B), set $X := \operatorname{Proj}(R)$ and let $\mathcal{F} = \widetilde{M}$ be the coherent sheaf induced by M. Then, it follows by (1.4)B)d),f) and (1.5)C) that for all $i \in \mathbb{N}_0$ we have

a) $p^{i+1}_M(n) = h^i_{\mathcal{F}}(n)$ for all $n \ll 0$.

So, in particular, for each $i \in \mathbb{N}_0$ there is a unique polynomial (namely p^{i+1}_M)

$$p^i_{\mathcal{F}} \in \mathbb{Q}[X] : \quad h^i_{\mathcal{F}}(n) = p^i_{\mathcal{F}}(n), \quad \forall n \ll 0,$$

the *i-th cohomoloical Serre polynomial of $\mathcal{F}$*. It follows easily that

b) $\deg(p^i_{\mathcal{F}}) \leq i$, with equality if $i = \dim(\mathcal{F}) \geq 0$.

c) If $i > \dim(\mathcal{F})$, then $p^i_{\mathcal{F}} = 0$.

d) $P_{\mathcal{F}} = \sum_{i=0}^{\dim(\mathcal{F})}(-1)^i p^i_{\mathcal{F}} = \sum_{i\in\mathbb{N}_0}(-1)^i p^i_{\mathcal{F}}$.

E) *(Cohomological Postulation Numbers of Sheaves)* Let the notations be as in part D). Then clearly

$$\nu^i_{\mathcal{F}} := \inf\{n \in \mathbb{Z} \mid p^i_{\mathcal{F}}(n) \neq h^i_{\mathcal{F}}(n)\} \in \mathbb{Z} \cup \{\infty\}.$$

The number $\nu^i_{\mathcal{F}}$ is called the *i-th cohomological postulation number of $\mathcal{F}$*. The following statements are easy to prove:

a) If $i \in \mathbb{N}$, then $\nu^i_{\mathcal{F}} = \nu^{i+1}_M$.

b) $\nu_{\mathcal{F}}^{0} \geq \min\{\nu_M^1, \mathrm{beg}(M)\}$.

c) If $\mathcal{F} \neq 0$, then $\nu_{\mathcal{F}}^{\dim(\mathcal{F})} \in \mathbb{Z}$.

d) If $i > \dim(\mathcal{F})$, then $\nu_{\mathcal{F}}^{i} = \infty$.

An important fact is the following observation.

3.8. **Proposition.** *Let $i \in \mathbb{N}_0$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then*

$$\nu_M^i \geq -\mathrm{reg}(K^i(M)).$$

Proof. This follows easily from the definition of ν_M^i, and (1.8)B)c). □

Next, we introduce the notion of *Canonical Module.*

3.9. **Definition.** Let K be a field and let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let $M \neq 0$ be a finitely generated graded R-module. We define the *canonical module $K(M)$ of M* as the highest order non vanishing module of deficiency of M, thus:

$$K(M) := K^{\dim_R(M)}(M).$$

Moreover we set $K(0) := 0$

Next, we aim to prove a basic result canonical modules. We begin with a statement on the *Grade of Canonical Modules.* This result already hints an important property of the operation of taking canonical modules: namely its "improving effect on grade". we start by recalling the notion of *Grade.*

3.10. **Reminder.** Let M be a finitely generated module over the Noetherian ring R and let $\mathfrak{a} \subseteq R$ be an ideal of R. Then the *grade*

$$\mathrm{grade}_M(\mathfrak{a})$$

of $\mathfrak{a}$ with respect to M is defined as the supremum of lengths r of M-sequences

$$x_1, x_2, \ldots, x_r \in \mathfrak{a}: \quad x_i \in \mathrm{NZD}_R(M/\sum_{j=1}^{i-1} x_j M, \quad \forall i \in \{1, 2, \ldots, r\},$$

in $\mathfrak{a}$. Keep in mind the well known fact (see [Br-Fu-Ro](4.4)(4.6) for example)

$$\mathrm{grade}_M(\mathfrak{a}) = \inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{a}}^i(M) \neq 0\}.$$

3.11. **Proposition.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra and let M de a finitely generated graded R-module. Then $\dim_R(M) = \dim_R(M)$ and moreover*

$$\mathrm{grade}_{K(M)}(R_+) \geq \min\{2, \dim_R(M)\}.$$

Proof. Let $d := \dim_R(M)$. If $d \leq 0$ our claim is obvious. So, let $d > 0$. By (3.7)e) we know that $K(M)$ is of dimension d. Now set $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$ and hence $K(\bar{M}) = K^d(\bar{M}) \cong K^d(M) = K(M)$ (see (3.4)A)c)). This allows us to replace M by $\bar{M}$ and hence to assume that $\Gamma_{R_+}(M) = 0$. So, by the Homogeneous Prime Avoidance Lemma we find some $t \in \mathbb{N}$ and some $x \in R_t \cap \mathrm{NZD}_R(M)$. Now, by the exact sequence (3.4)B)b), applied with $i = d$, we get an epimorphism

$$K^d(M/xM) \to (0 :_{K^d(M)} x) \to 0.$$

As $x \in R_+ \cap \mathrm{NZD}_R(M)$ we also have $\dim_R(M/xM) = d - 1$ and hence $K^d(M/xM) = 0$ (see ((3.7)d)). It follows that $(0 :_{K^d(M)} x) = 0$ and hence $x \in \mathrm{NZD}_R(K^d(M))$. Thus, if $d = 1$, we get our claim. So, let $d > 1$. Another use of the sequence (3.4)B)b), this time applied with $i = d - 1$, yields a monomorphism

$$0 \to \big(K^d(M)/xK^d(M)\big)(t) \to K^{d-1}(M/xM).$$

As $\dim_R(M/xM) = d-1 > 0$ we have $K^{d-1}(M/xM) = K(M/xM)$ and hence by induction we get $\mathrm{grade}_{K^{d-1}(M/xM)}(R_+) > 0$, hence $\Gamma_{R_+}(M/xM) = 0$. Now, the above monomorphism shows that $\Gamma_{R_+}\big((K^d(M)/xK^d(M))(t)\big) = 0$ and hence $\Gamma_{R_+}(K^d(M)/xK^d(M)) = 0$, so that $\mathrm{grade}_{K^d(M)/xK^d(M)}(R_+) \geq 1$. As $x \in R_+ \cap \mathrm{NZD}_R(K^d(M))$ it follows that $\mathrm{grade}_{K^d(M)}(R_+) \geq 2$ and this proves our claim. $\square$

4. Regularity of Modules of Deficiency

Already in Mumfords Lecture Notes [Mu1] the study of the regularity of deficiency modules is called to be of basic significance. In this section, we are precisely concerned with this issue. Our main result will say that the regularity of the deficiency modules of a given finitely generated graded module over a Noetherian homogeneous K-algebra is bounded in terms of the cohomology diagonal of M and the beginning of M. We rephrase this a bit more precisely: Let $d \in \mathbb{N}$ and let $i \in \mathbb{N}_0$. Then, there is a function

$$G_d^i : \mathbb{N}_0^d \times \mathbb{Z} \to \mathbb{Z}$$

such that for each pair $(R, M) \in \mathcal{M}^d$ we have the estimate

$$\operatorname{reg}\big(K^i(M)\big) \leq G_d^i\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

We begin with some preparations. First we recall the Notion of *Filter-Regular Element.*

4.1. **Definition and Remark.** A) *(Filter-Regular Elements)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Let $t \in \mathbb{N}$ and let $x \in R_t$. Then x is said to be *filter-regular with respect to M* if the following equivalent conditions are satisfied:

(i) $x \in \mathrm{NZD}_R(M/\Gamma_{R+}(M))$.

(ii) $x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Proj}(R)}$.

(iii) $(0 :_R x) \subseteq \Gamma_{R_+}(M)$.

(iv) $\operatorname{end}(0 :_R x) < \infty$.

(v) The multiplication map $x : M_n \to M_{n+1}$ is injective for all $n \gg 0$.

B) *(Existence of Filter-Regular Elements)* Let the hypotheses and notations be as in part A). Then, on use of the *Homogeneous Prime Avoidance Priciple* we can say:

a) There is some $t_0 \in \mathbb{N}$ such that for each integer $t \geq t_0$ there is an element $x \in R_t$ which is filter-regular with respect to M.

b) If K is infinite, the number t_0 of statement a) may be chosen to be finite.

Here comes a first result about filter-regular elements.

4.2. **Lemma.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module and let $x \in R_1$ be a filter-regular element with respect to M. Then*

$$\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq \operatorname{reg}(M).$$

Proof. We have two short exact sequences of graded R-modules

$$0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0,$$

$$0 \to (M/(0 :_M x))(-1) \to M \to M/xM \to 0.$$

As $(0 :_M x)$ is R_+-torsion we get an isomorphism of graded R-modules

$$H^1_{R_+}(M) \cong H^1_{R_+}(M/(0 :_M x)),$$

so that $\operatorname{reg}^1(M/(0 :_M x)) = \operatorname{reg}^1(M)$. Now, if we apply cohomology to the second exact sequence it follows by (2.2) A)c) that

$$\operatorname{reg}^1(M) = \operatorname{reg}^1\big(M/(0 :_M x)\big) = \operatorname{reg}\big((M/(0 :_M x))(-1)\big) - 1 \leq$$

$$\leq \max\{\operatorname{reg}^1(M), \operatorname{reg}(M/xM) + 1\} - 1,$$

whence $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM)$.

By another application of cohomology to the second sequence we similarly get

$$\operatorname{reg}(M/xM) \leq \max\{\operatorname{reg}^1\big((M/(0 :_M x))(-1)\big) - 1, \operatorname{reg}(M)\} =$$

$$= \max\{\operatorname{reg}^1(M), \operatorname{reg}(M)\} = \operatorname{reg}(M),$$

whence $\operatorname{reg}(M/xM) \leq \operatorname{reg}(M)$. □

Here comes a first application of the previous lemma, which will be of use in the proof ot the main result of this section.

4.3. **Proposition.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $x \in R_1$ be filter-regular with respect to M and let $m \in \mathbb{Z}$ be such that* $\operatorname{reg}(M/xM) \leq m$ *and* $\operatorname{gendeg}((0 :_M x)) \leq m$. *Then*

$$\operatorname{reg}(M) \leq m + h^0_M(m).$$

Proof. By (4.2) we have $\operatorname{reg}^1(M) \leq \operatorname{reg}(M/xM) \leq m$. So, it remains to show that

$$\operatorname{end}(H^0_{R_+}(M)) \leq m + h^0_M(m).$$

The short exact sequence of graded R-modules

$$0 \to (M/(0 :_M x))(-1) \to M \to M/xM \to 0$$

induces exact sequences of K-vector spaces

$$0 \to H^0_{R_+}(M/(0 :_M x))_n \to H^0_{R_+}(M)_{n+1} \to$$

$$\to H^0_{R_+}(M/xM)_{n+1} \to H^1_{R_+}(M/(0 :_M x))_n$$

for all $n \in \mathbb{Z}$. As $H^0_{R_+}(M/xM)_{n+1} = 0$ for all $n \geq m$, we therefore obtain

$$H^0_{R_+}(M/(0 :_M x))_n \cong H^0_{R_+}(M)_{n+1}, \quad \forall n \geq m.$$

The short exact sequence of graded R-modules

$$0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0$$

and the facts that

$$H^0_{R_+}\big((0 :_M x)\big) = (0 :_M x), \quad H^1_{R_+}\big((0 :_M x)\big) = 0$$

induces short exact sequences of K-vector spaces

$$0 \to (0 :_M x)_n \to H^0_{R_+}(M)_n \to H^0_{R_+}(M/(0 :_M x))_n \to 0, \quad \forall n \in \mathbb{Z}.$$

So, for all $n \geq m$ we get an exact sequence of K-vector spaces

$$0 \to (0 :_M x)_n \to H^0_{R_+}(M)_n \xrightarrow{\pi_n} H^0_{R_+}(M)_{n+1} \to 0.$$

To prove our claim we may assume that $\mathrm{end}(H^0_{R_+}(M)) > m$. As

$$\mathrm{end}((0 :_M x)) = \mathrm{end}\big(H^0_{R_+}(M)\big), \quad \mathrm{gendeg}((0 :_M x)) \leq m$$

it follows that

$$(0 :_M x)_n \neq 0, \quad \forall n \in \{m, m+1, \ldots, \mathrm{end}(H^0_{R_+}(M))\}.$$

Hence for all these values of n the homomorphism π_n is surjective but not injective. Therefore

$$h^0_M(n) > h^0_M(n+1), \quad \forall n \in \{m, m+1, \ldots, \mathrm{end}(H^0_{R_+}(M))\}.$$

So, in the range $n \geq m$ the function $n \mapsto h^0_M(n)$ is strictly decreasing until it reaches the value 0. Therefore $h^0_M(n) = 0$ for all $n > m + h^0_M(m)$. This proves our claim. $\square$

In the proof of our main result we have to perform a number of induction arguments, which use filter-regular elements of degree 1. In general, such elements only exist if the base field K of our Noetherian homogeneous ring R is infinite (see (4.2)B)b)). So, we must be able to replace R by an appropriate Noetherian homogeneous algebra over an infinite field. The following remark is aimed to prepare this. For a more detailed presentation of the subjet we refer to [Br8] (10.7).

4.4. **Remark.** A) *(Base Field Extensions of Homogeneous Algebras)* Let K be a field, let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a Noetherian homogeneous K-algebra and let K' be an extension field of K. Then, the K' algebra $R' := K' \otimes_K R$ carries a natural grading, given by

$$R' = K' \otimes_K R = K' \oplus (K' \otimes_K R_1) \oplus (K' \otimes_K R_2) \oplus \ldots,$$

which turns R' into a Noetherian homogeneous K'-algebra with irrelevant ideal $R'_+ = R_+R'$.

B) *(Base Field Extensions and Graded Modules)* Let the notations and hypotheses be as in part A) and assume in addition, that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded R-module. Then, the R'-module $M' := R' \otimes_R M = K' \otimes_K M$ carries a natural grading given by

$$M' = R' \otimes_R M = K' \otimes_K M = \bigoplus_{n\in\mathbb{Z}} K' \otimes_K M_n.$$

Moreover we can say:

a) $\mathrm{gendeg}(M') = \mathrm{gendeg}(M)$.

b) $\mathrm{beg}(M') = \mathrm{beg}(M)$.

c) $\mathrm{end}(M') = \mathrm{end}(M)$.
d) $\dim'_K(M'_n) = \dim_K(M_n)$ for all $n \in \mathbb{Z}$.
e) M' is finitely generated over R' if and only if M is finitely generated.

C) *(Base Field Extensions and Local Cohomology)* Keep the notations and hypotheses of parts A) and B). Then, the *Graded Flat Base Change Property of Local Cohomology* (see [Br-Sh1] (13.1.8) or [Br8] (1.15)) gives rise to isomorphisms of graded R'-modules

$$H^i_{R'_+}(M') \cong R' \otimes_R H^i_{R_+}(M) = K' \otimes_K H^i_{R_+}(M), \quad \forall i \in N_0.$$

If the graded R-module M is finitely generated, we thus can say (see statements a)-e) of part B)):

a) $h^i_{M'}(n) = h^i_M(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
b) $d^i_{M'}(n) = d^i_M(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.
c) $\mathrm{reg}^l(M') = \mathrm{reg}^l(M)$ for all $l \in \mathbb{N}_0$.

D) *(Base Field Extensions and Deficiency Modules)* Keep the above notations and hypotheses. Then by the fact that taking vector-space duals naturally commutes with field extensions the observations made in part C) imply that there are isomorphisms of graded R-modules

$$K^i(M') \cong R' \otimes_R K^i(M) = K' \otimes_K K^i(M), \quad \forall i \in \mathbb{N}_0.$$

Consequently, for each finitely generated graded R-module M statement b) of part C) implies

a) $\mathrm{reg}(K^i(M')) = \mathrm{reg}(K^i(M))$ for all $i \in \mathbb{N}_0$.

4.5. **Lemma.** *Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module. Then, for all $i \in \mathbb{N}_0$ and all $n \geq\ i$ we have*

$$\dim_K\big(K^{i+1}(M)_n\big) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \big[\sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}_M(l-i)\big].$$

Proof. We only sketch this proof. For more deatails see [Br8] (10.4). If K is finite, we chose an infinite extension field K' of K. Then, the observations made in (4.4) allow to replace R and M resectively by $R' := K' \otimes_K R$ and $M' := R' \otimes_R M = K' \otimes_K M$. So we may assume at once that the base field is infinite. Our first aim is to show the following statement (see [Br8](8.12)):

a) $d^i_M(-n) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \big[\sum_{l=0}^{i-j} \binom{i-j}{l} d^{i-l}_M(l-i)\big]$, $\forall i \in \mathbb{N}_0 \forall n \geq i$.

By (4.1)B)b) we find an element $x \in R_1$ which is filter-regular with respect to M. In view of the natural isomorphisms of graded R-modules $D^i_{R_+}(M) \cong D^i_{R_+}(M/\Gamma_{R_+}(M))$ for all $i \in \mathbb{N}_0$ we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume the $x \in \mathrm{NZD}(M)$. The right derived sequence of the funtcor $D_{R+}(\bullet)$ associated to the short exact sequence of graded R-modules

$$\mathbb{S}: \quad 0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0$$

gives rise to a monomorphism of graded R-modules

$$0 \to D^0_{R_+}(M)(-1) \xrightarrow{x} D^0 R_+(M)$$

and exact sequences of graded R-modules

$$D^{i-1}_{R_+}(M) \to D^{i-1}_{R_+}(M/xM) \to D^i_{R_+}(M)(-1) \xrightarrow{x} D^i_{R_+}(M), \quad \forall i \in \mathbb{N}.$$

Consequntly we get

b) $d^0_M(-n) \leq d^0_M(-(n-1)), \quad \forall n \in \mathbb{Z}$

and $d^i_M(-n) \leq d^i_M(-(n-1)) + d^{i-1}_{M/xM}(-(n-1))$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$ and hence

c) $d^i_M(-n) \leq d^i_M(-i) + \sum_{m=i-1}^{n-1} d^{i-1}_{M/xM}(-m), \quad \forall i \in \mathbb{N}, \forall n \in \mathbb{Z}.$

d) $d^j_{M/xM}(-j) \leq d^j_M(-j) + d^{j+1}_M(-(j+1)), \quad \forall j \in \mathbb{N}_0.$

Now, inequality b) proves statment a) if $i = 0$. The inequalities c) and d) together with the Pascal equalities for binomial coefficients allow to prove statement a) by induction on i.

Now, we show that statement a) implies our lemma. If $i > 0$, by (1.4)B)f) we have

$$d^i_M(-n) = h^{i+1}_M(-n).$$

Moreover $h^0_M(-n) \leq \dim_K(M_n)$, whence

$$h^1_M(-n) \leq \dim_K(M_{-n}) - h^0_M(-n) + h^1_M(-n) = d^0_M(-n).$$

As $h^{i+1}_M(-n) = \dim_K(K^{i+1}(M)_n)$ (see (3.6)b)) our claim follows. $\square$

Now, we define the bounding functions $G^i_d : N_0 \times \mathbb{Z} \to \mathbb{Z}$, which were mentioned already at the beginning of this section.

4.6. **Definition.** *(A Class of Bounding Functions)* For all $d \in \mathbb{N}$ and all $i \in \{0, 1, \ldots, d\}$ we define the functions

$$G^i_d : \mathbb{N}^d_0 \times \mathbb{Z} \to \mathbb{Z}$$

recursevely as follows. In the case $i = 0$ we define

(i) $\quad G^0_d(x_0, x_1, \ldots, x_{d-1}, y) := -y.$

In the case $i = 1$ we set:

(ii) $\quad G^1_1(x_0, y) := y - 1;$

(iii) $\quad G^1_d(x_0, x_1, \ldots, x_{d-1}, y) := \max\{0, 1-y\} + \sum_{i=0}^{d-2} \binom{d-1}{i} x_{d-i-2}$, if $d \geq 2$.

In the case $i = d = 2$ we define

(iv) $G_2^2(x_0, x_1, y) := G_2^1(x_0, x_1, y) + 2$.

Now, assume that $d \geq 3$ and that the functions $G_{d-1}^{i-1}, G_{d-1}^{i}$ and G_d^{i-1} are already defined. In order to define the function G_d^i we first intermediately introduce the following notation:

(v) $m_i := \max\{G_{d-1}^{i-1}(x_0+x_1, \ldots, x_{d-2}+x_{d-1}, y), G_d^{i-1}(x_0, \ldots, x_{d-1}, y)+1\}+1$.
(vi) $n_i := G_{d-1}^{i}(x_0 + x_1, ..., x_{d-2} + x_{d-1}, y)$,
(vii) $t_i := \max\{m_i, n_i\}$,
(viii) $\Delta_{ij} := \sum_{l=0}^{i-j-1} \binom{i-j-1}{l} x_{i-l-1}$.

Using these notational conventions, we define

(ix) $G_d^i(x_0, \ldots, x_{d-1}, y) := t_i + \sum_{j=0}^{i-1} \Delta_{ij}, \quad \forall i \in \{2, 3, \ldots, d-1\}$.

Finally, if $d \geq 3$ and G_{d-1}^{d-1} and G_d^{d-1} are already defined, we set (see (v))

(x) $G_d^d(x_0, \ldots, x_{d-1}, y) := m_d$.

In order to prove our main result, we need a particular property of the previously defined bounding function

4.7. **Exercise.** *(Monotonicity of the Bounding Functions G_d^i)* Let $d \in \mathbb{N}_0$, let $i \in \{0, 1, \ldots, d\}$ and let

$$(x_0, x_1, \ldots, x_{d-1}, y), \quad (x'_0, x'_1, \ldots, x'_{d-1}, y') \in \mathbb{N}_0^d \times \mathbb{Z}$$

such that

$$x_j \leq x'_j, \quad \forall j \in \{0, 1, \ldots, d-1\}, \quad y' \leq y.$$

Prove by induction on i and d, that under these circumstances we have

$$G_d^i(x_0, x_1, \ldots, x_{d-1}, y) \leq G_d^i(x'_0, x'_1, \ldots, x'_{d-1}, y').$$

The following exercise generalizes an argument made in the proof of (4.5).

4.8. **Exercise.** Let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra, let M be a finitely generated graded R-module, let $t \in \mathbb{N}$ and let $x \in R_t$ be filter-regular with respect to M. Show that for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we have the inequality

$$d_{M/xM}^i(n) \leq d_M^i(n) + d_M^{i+1}(n-t).$$

Now, we are ready to formulate and to prove the announced main result.

4.9. **Theorem.** *Let $d \in \mathbb{N}$, let $i \in \{0, 1, \ldots, d\}$, let K be a field, let $R = K \oplus R_1 \oplus R_2 \ldots$ be a Noetherian homogeneous K-algebra and let M be a finitely generated graded R-module with $\dim_R(M) = d$. Then*

$$\operatorname{reg}\big(K^i(M)\big) \leq G_d^i\big(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \operatorname{beg}(M)\big).$$

Proof. We proceed by induction on i. By (3.6)e) we have $\dim_R(K^0(M)) \leq 0$. So, in view of (3.6)b) we get

$$\operatorname{reg}(K^0(M)) = \operatorname{end}(K^0(M)) = -\operatorname{beg}(H^0_{R_+}(M)) \leq -\operatorname{beg}(M) =$$
$$G^0_d\big(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\big).$$

This clearly proves the case $i = 0$.

So let $i > 0$. As in the proof of (4.5) we may use the observations made in (4.4) to assume that K is infinite.

Let $\bar{M} := M/\Gamma_{R_+}(M)$. Then $\dim_R(\bar{M}) = d$, and in view of the natural isomorphisms of graded R-modules $D^i_{R_+}(M) \cong D^i_{R_+}(\bar{M})$ we have $d^j_{\bar{M}}(n) = d^j_M(n)$ for all $j \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. In addition $\operatorname{beg}(M) \leq \operatorname{beg}(\bar{M})$, whence by (4.7) we get

$$G^i_d\big(d^0_{\bar{M}}(0), d^1_{\bar{M}}(-1) \ldots, d^{d-1}_{\bar{M}}(1-d), \operatorname{beg}(\bar{M})\big) \leq$$
$$\leq G^i_d\big(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\big).$$

As moreover we have an isomorphism of graded R-modules $K^i(\bar{M}) \cong K^i(M)$ (see (3.4)A)c)), we thus may replace M by $\bar{M}$ and hence assume that $\Gamma_{R_+}(M) = 0$. Therefore we find some element $x \in R_1 \cap \operatorname{NZD}_R(M)$. By Homogeneous Prime Avoidance we may assume in addition, that x is filter-regular with respect to the modules $K^0(M), K^1(M), \ldots K^d(M)$. By (3.4)B)b) there is an exact sequence of graded R-modules

a) $0 \to \big(K^{j+1}(M)/xK^{j+1}(M)\big)(+1) \to K^j(M/xM) \to (0 :_{K^j(M)} x) \to 0,$

for all $j \in \mathbb{N}_0$. Since $H^0_{R_+}(M) = 0$ we have $K^0(M) = 0$ (see (3.6)b) for example), so that the sequence a) gives rise to an isomorphism of graded R-modules

b) $\big(K^1(M)/xK^1(M)\big)(+1) \cong K^0(M/xM).$

As $\dim_R(K^0(M/xM)) \leq 0$ (see (3.6)e)), the above isomorphism shows that $K^1(M)/xK^1(M)$ is R_+-torsion, so that (see (3.6)b))

$$\operatorname{reg}\big(K^1(M)/xK^1(M)\big) = \operatorname{reg}\big(K^0(M/xM)\big) + 1 = \operatorname{end}\big(K^0(M/xM)\big) + 1 =$$
$$1 - \operatorname{beg}\big(H^0_{R_+}(M/xM)\big) \leq 1 - \operatorname{beg}(M/xM) \leq 1 - \operatorname{beg}(M).$$

It follows that

c) $\operatorname{reg}\big(K^1(M)/xK^1(M)\big) \leq 1 - \operatorname{beg}(M).$

We first assume that $d = 1$. Then clearly $i = 1$, whence $K^i(M) = K^1(M) = K(M)$ so that by (3.11) we get $\operatorname{grade}_{K^1(M)}(R_+) = 1$ hence $H^0_{R_+}(K^1(M)) = 0$, so that $\operatorname{reg}(K^1(M)) = \operatorname{reg}^1(K^1(M))$. It follows that (see (4.2))

$$\operatorname{reg}\big(K^1(M)\big) \leq \operatorname{reg}\big(K^1(M)/xK^1(M)\big) \leq 1 - \operatorname{beg}(M) = G^1_1(d^0_M(0), \operatorname{beg}(M)\big).$$

This proves our claim if $d = 1$.

So, assume from now on, that $d \geq 2$. We first treat the case $i = 1$. To do so, we consider the sequence a) for $j = 1$, hence

d) $0 \to \big(K^2(M)/xK^2(M)\big)(+1) \to K^1(M/xM) \to (0 :_{K^1(M)} x) \to 0.$

If $d = 2$, we have $\dim_R(M/xM) = 1$ and so by the aleady treated case $d = 1$ we get

$$\operatorname{reg}\big(K^1(M/xM)\big) \leq 1 - \operatorname{beg}(M/xM) \leq 1 - \operatorname{beg}(M).$$

Consequently by (2.2)A)e) we have

$$\operatorname{gendeg}\big((0 :_{K^1(M)} x)\big) \leq \operatorname{gendeg}\big(K^1(M/xM)\big) \leq$$
$$\leq \operatorname{reg}\big(K^1(M/xM)\big) \leq 1 - \operatorname{beg}(M).$$

Assume first that $m_0 := 1 - \operatorname{beg}(M) \leq 0$. Then, by (4.3) (applied with $m = 0$) we obtain (see (3.6)b))

$$\operatorname{reg}\big(K^1(M)\big) \leq 0 + h^0_{K^1(M)}(0) \leq \dim_K\big(K^1(M)_0\big) = h^1_M(0) \leq d^0_M(0).$$

Now, assume that $m_0 := 1 - \operatorname{beg}(M) > 0$. Then $d^0_M(-m_0) \leq d^0_M(0)$ (see statement b) in the proof of (4.5)). So by statement c), by (4.3) and by (3.6)b) we get

$$\operatorname{reg}\big(K^1(M)\big) \leq m_0 + h^0_{K^1(M)}(m_0) \leq m_0 + \dim_K(K^1(M)_{m_0}) =$$
$$= 1 - \operatorname{beg}(M) + h^1_M(-m_0) \leq 1 - \operatorname{beg}(M) + d^0_M(-m_0) \leq 1 - \operatorname{beg}(M) + d^0_M(0).$$

Therefore, bearing in mind (4.6)(iii) we finally obtain

$$\operatorname{reg}\big(K^1(M)\big) \leq \max\{d^0_M(0), 1 - \operatorname{beg}(M) + d^0_M(0)\} \leq$$
$$\leq \max\{0, 1 - \operatorname{beg}(M)\} + d^0_M(0) = G^1_2\big(d^0_M(0), d^1_M(-1), \operatorname{beg}(M)\big).$$

This proves the case in which $d = 2$ and $i = 1$.

Now, let $d \geq 3$, but still let $i = 1$. Then, by induction on d we may write (see (4.6)(iii))

$$\operatorname{reg}\big(K^1(M/xM)\big) \leq G^1_{d-1}\big(d^0_{M/xM}(0), \ldots, d^{d-2}_{M/xM}(2-d), \operatorname{beg}(M/xM)\big) =$$
$$= \max\{0, 1 - \operatorname{beg}(M/xM)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} d^{d-i-3}_{M/xM}(i+3-d).$$

According to (4.8) we have

$$d^{d-i-3}_{M/xM}(i+3-d) \leq d^{d-i-3}_M(i+3-d) + d^{d-i-2}_M(i+2-d), \quad \forall i \in \{0, 1, \ldots, d-3\}.$$

Therefore we obtain

$$\operatorname{reg}\big(K^1(M/xM)\big) \leq$$
$$\leq \max\{0, 1 - \operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} \big[d^{d-i-3}_M(i+3-d) + d^{d-i-2}_M(i+2-d)\big] =: t_0.$$

By the exact sequence d) and (2.2)A)e) we now get

$$\operatorname{gendeg}\big((0 :_{K^1(M)} x)\big) \leq \operatorname{reg}\big(K^1(M/xM)\big) \leq t_0.$$

By the above inequality c) and the definition of t_0 we have

$$\operatorname{reg}\big(K^1(M)/xK^1(M)\big) \leq t_0.$$

As $t_0 \geq 0$ we also have $d^0_M(-t_0) \leq d^0_M(0)$. So, by (4.3) and (3.6)b) we obtain the inequalities

$$\operatorname{reg}\big(K^1(M)\big) \leq t_0 + h^0_{K^1(M)}(t_0) \leq t_0 + \dim_K\big(K^1(M)_{t_0}\big) =$$

$$= t_0 + h^1_M(-t_0) \leq t_0 + d^0_M(-t_0) \leq t_0 + d^0_M(0) =$$

$$= \max\{0, 1-\operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} \big[d_M^{d-i-3}(i+3-d) + d_M^{d-i-2}(i+2-d)\big] + d^0_M(0) =$$

$$= \max\{0, 1-\operatorname{beg}(M)\} + d_M^{d-2}(2-d) + \sum_{i=1}^{d-3} \Big[\binom{d-2}{i-1} + \binom{d-2}{i}\Big] d_M^{d-i-2}(i+2-d) +$$

$$+ (d-2)d^0_M(0) + d^0_M(0) =$$

$$= \max\{0, 1-\operatorname{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-1}{i} d_M^{d-i-2}(i+2-d) + (d-1)d^0_M(0) =$$

$$= \max\{0, 1-\operatorname{beg}(M)\} + \sum_{i=0}^{d-2} \binom{d-1}{i} d_M^{d-i-2}(i+2-d).$$

In view of (4.6)(iii) this means that

$$\operatorname{reg}\big(K^1(M)\big) \leq G^1_d\big(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\big).$$

So, we have settled the case $i = 1$ for all $d \in \mathbb{N}$.

We now attack the cases with $i \geq 2$. We begin with the case in which $d = 2$ and hence $i = 2$. In view of the exact sequence d) we obtain (see (2.2)A)c),f))

$$\operatorname{reg}\big(K^2(M)/xK^2(M)\big) \leq \max\{\operatorname{reg}\big(K^1(M/xM)\big), \operatorname{reg}\big((0 :_{K^1(M)} x)\big) + 1\} + 1.$$

Observe that $\dim_R(M/xM) = 1$, so that by what we know from the already treated case $i = d = 1$ we get

$$\operatorname{reg}\big(K^1(M/xM)\big) \leq G^1_1\big(d^0_{M/xM}(0), \operatorname{beg}(M/xM)\big) =$$

$$\operatorname{beg}(M/xM) - 1 \leq \operatorname{beg}(M) - 1.$$

As x is filter-regular with respect to $K^1(M)$, we have $(0 :_{K^1(M)} x) \subseteq H^0_{R_+}(M)$, so that

$$\operatorname{reg}\big((0 :_{K^1(M)} x)\big) = \operatorname{end}\big((0 :_{K^1(M)} x)\big) \leq \operatorname{end}\big(H^0_{R_+}(K^1(M))\big) \leq \operatorname{reg}\big(K^1(M)\big).$$

By what we know from the already treated case with $i = 1$ and $d = 2$ we have

$$\operatorname{reg}\big(K^1(M)\big) \leq G^1_2\big(d^0_M(0), d^1_M(-1), \operatorname{beg}(M)\big) = \max\{0, 1-\operatorname{beg}(M)\} + d^0_M(0).$$

Therfore we get

$$\operatorname{reg}\big(K^2(M)/xK^2(M)\big) \leq \max\{1-\operatorname{beg}(M), \max\{0, 1-beg(M)\} + d^0_M(0) + 1\} + 1$$

$$\leq \max\{0, 1-\operatorname{beg}(M)\} + d^0_M(0) + 2.$$

As $\operatorname{grade}_{K^2(M)}(R_+) = \operatorname{grade}_{K(M)}(R_+) \geq \min\{2, d\} = 2 = d$ (see (3.11)) we have $\operatorname{grade}_{K^2(M)}(R_+) = 2$, whence $H^j_{R_+}(K^2(M)) = 0$ for $j = 0, 1$. This means that $\operatorname{reg}(K^2(M)) = \operatorname{reg}^1(K^2(M))$. So by(4.2) we obtain

$$\operatorname{reg}\bigl(K^2(M)\bigr) \leq \operatorname{reg}\bigl(K^2(M)/xK^2(M)\bigr) \leq$$

$$\leq \max\{0, 1 - \operatorname{beg}(M)\} + d^0_M(0) + 2 = G^2_2\bigl(d^0_M(0), d^1_M(-1), \operatorname{beg}(M)\bigr).$$

This completes our proof in the cases with $i \geq 2$ and $d = 2$.

So, let $d > 2$ and $i \geq 2$. By (4.8) we have

$$d^j_{M/xM}(-j) \leq d^j_M(-j) + d^{j+1}_M(-j-1), \quad \forall j \in \mathbb{N}_0.$$

Let $k \in \{0, 1, \ldots, d-1\}$. Then, by induction on d and in view of (4.7) we have

$$\operatorname{reg}\bigl(K^k(M/xM)\bigr) \leq G^k_{d-1}\bigl(d^0_{M/xM}(0), \ldots, d^{d-2}_{M/xM}(2-d), \operatorname{beg}(M/xM)\bigr) \leq$$

$$\leq G^k_{d-1}\bigl(d^0_M(0) + d^1_M(-1), \ldots, d^{d-2}_M(2-d) + d^{d-1}_M(1-d), \operatorname{beg}(M)\bigr) =: n_k.$$

Therefore

e) $\operatorname{reg}\bigl(K^k(M/xM)\bigr) \leq n_k$ for all $k \in \{0, 1, \ldots, d-1\}$.

Clearly, by induction on i we have

f) $\operatorname{reg}\bigl(K^{i-1}(M)\bigr) \leq G^{i-1}_d\bigl(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\bigr) =: v_{i-1}$.

If we apply the exact sequence a) with $j = i - 1$ we get (see (2.2)A)c),f))

$\operatorname{reg}\bigl(K^i(M)/xK^i(M)\bigr) \leq \max\{\operatorname{reg}\bigl(K^{i-1}(M/xM)\bigr), \operatorname{reg}\bigl((0 :_{K^{i-1}(M)} x)\bigr) + 1\} + 1.$

By the inequality e) we have

$$\operatorname{reg}\bigl(K^{i-1}(M/xM)\bigr) \leq n_{i-1}.$$

Moreover, as x is filter-regular with respect to $K^{i-1}(M)$ we have once more $\operatorname{reg}\bigl((0 :_{K^{i-1}(M)} x)\bigr) \leq \operatorname{end}\bigl(H^0_{R_+}(K^{i-1}(M))\bigr) \leq \operatorname{reg}\bigl(K^{i-1}(M)\bigr)$, so that by the inequality f) we have

$$\operatorname{reg}\bigl((0 :_{K^{i-1}(M)} x)\bigr) \leq v_{i-1}.$$

Thus, gathering together we we obtain (see (4.6)(v)):

g) $\operatorname{reg}\bigl(K^i(M)/xK^i(M)\bigr) \leq \max\{n_{i-1}, v_{i-1} + 1\} + 1 = m_i$.

Assume first, that $2 \leq i \leq d - 1$. By the definitions of v_i and n_i (see (4.6)(v),(vi)) it follows easily

$$t_i = \max\{m_i, n_i\} \geq i.$$

Moreover, if we apply the sequence a) with $j = i$ and keep in mind the inequality e) we get (see also (2.2)A)e))

$$\operatorname{gendeg}\bigl((0 :_{K^i(M)} x)\bigr) \leq \operatorname{reg}\bigl(K^i(M/xM)\bigr) \leq n_i.$$

So, by (4.3), applied to the graded R-module $K^i(M)$ with $m := t_i$ and with (4.5) applied with $n = t_i$ and with $i-1$ instead of i we obtain

$$\operatorname{reg}\bigl(K^i(M)\bigr) \leq t_i + h^0_{K^i(M)}(t_i) \leq t_i + \dim_K\bigl(K^i(M)_{t_i}\bigr) \leq$$

$$\leq t_i + \sum_{j=0}^{i-1} \binom{t_i - j - 1}{i - j - 1} \bigl[\sum_{l=0}^{i-j-1} \binom{i-j-1}{l} d_M^{i-l-1}(l-i+1) \bigr].$$

In view of (4.6)(viii),(ix) this means that

$$\operatorname{reg}\bigl(K^i(M)\bigr) \leq G^i_d\bigl(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\bigr).$$

This completes our proof in the cases with $i \leq d-1$.

It remains to treat the cases with $i = d > 2$. Observe that by (3.11) we have $\operatorname{grade}_{K^d(M)}(R_+) = 2$, so that again $\operatorname{reg}\bigl(K^d(M)\bigr) = \operatorname{reg}^1\bigl(K^d(M)\bigr)$. Keep in mind, that x is filter-regular with respect to $K^d(M)$. So, if we apply (4.2) to this latter module and bear in mind the previous inequality g) we obtain

$$\operatorname{reg}\bigl(K^d(M)\bigr) \leq \operatorname{reg}\bigl(K^d(M)/xK^d(M)\bigr) \leq m_d.$$

In view of (4.6)(x) this means that

$$\operatorname{reg}\bigl(K^d(M)\bigr) \leq G^d_d\bigl(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \operatorname{beg}(M)\bigr).$$

This completes our proof. $\square$

4.10. **Corollary.** *Let $d \in \mathbb{N}$, and let $x_0, x_1, \ldots, x_{d-1} \in \mathbb{N}_0$ and $y \in \mathbb{Z}$. Then for each pair $(R, M) \in \mathcal{M}^d$ such that*

$$d^j_M(-j) \leq x_j \quad \forall j \in \{0, 1, \ldots, d-1\}, \quad \operatorname{beg}(M) \geq y$$

it holds

$$\operatorname{reg}\bigl(K^i(M)\bigr) \leq G^i_d(x_0, x_1, \ldots, x_{d-1}, y), \quad \forall i \in \{0, 1, \ldots, d\}.$$

Proof. This is immediate by (4.9) and (4.7). $\square$

4.11. **Remark.** *(Around Regularity of Deficiency Modules)* (see [Br-Ja-Li1]) The main result Theorem 4.9 of the present section and its consequence Corollary 4.10 may be proved in a more general context. These results namely hold over all Noetherian homogeneous rings $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ with Artinian local base Ring R_0. Clearly, in this general setting, the notion of deficiency module has to be defined in a different way. Moreover, these results have a number of further applications, which we quote here in simplified form. We always assume that R is as above, that M is a finitely generated graded R-module and that $i \in \mathbb{N}_0$.

a) The invariant $\operatorname{reg}(K^i(M))$ is bounded in terms of the three invvariants $\operatorname{beg}(M)$, $\operatorname{reg}^2(M)$ and $P_M(\operatorname{reg}^2(M))$.
b) If $\mathfrak{a} \subseteq R$ is a graded ideal, then $\operatorname{reg}(K^i(\mathfrak{a}))$ and $\operatorname{reg}(K^i(R/\mathfrak{a}))$ can be bounded in terms of the three invariants $\operatorname{reg}^2(\mathfrak{a})$, $\operatorname{length}(R_0)$, $\operatorname{reg}^1(R)$ and the number of generating one-forms of R.

In the particular case, where $R = R_0[X_1, X_2, \ldots, X_d]$ is a standard graded polynomial ring over the Artinian local ring R_0, we have in addition the following statements.

c) If $U \neq 0$ is a finitely generated graded R-module and $M \subseteq U$ is a graded submodule, then $\operatorname{reg}(K^i(M))$ and $\operatorname{reg}(K^i(U/M))$ are bounded in terms of d, $\operatorname{length}(R_0)$, $\operatorname{beg}(U)$, $\operatorname{reg}(U)$, the number of generators of U and the generating degree $\operatorname{gendeg}(U)$ of U.
d) If U and M are as in statement c), then $\operatorname{reg}(K^i(M))$ and $\operatorname{reg}(K^i(U/M))$ are bounded in terms of $\operatorname{length}(R_0)$, $\operatorname{beg}(U)$, $\operatorname{reg}(U)$, the Hilbert polynomial P_U of U and the Hilbert polynomial P_M of M.
e) It $p : F \to M \to 0$ is an epimorphism of graded R-modules such that F is free and of finite rank r, then $\operatorname{reg}(K^i(M))$ is bounded in terms of d, r, $\operatorname{length}(R_0)$, $\operatorname{beg}(F)$, $\operatorname{gendeg}(F)$ and $\operatorname{gendeg}(\operatorname{Ker}(p))$.

5. Bounding Cohomology

Now, we return to our original aim, which was to give criteria allowing to specify classes of finite cohomology. We begin with a bound for the cohomological postulation numbers

$$\nu^i_M := \inf\{n \in \mathbb{Z} \mid p^i_M(n) \neq h^i_M(n)\}$$

of a finitely generated graded module M over a Noetherian homogeneous K-algebra R, as they were introduced in (3.7)C). The notations are the same as in Section 4.

5.1. **Corollary.** *Let $d \in \mathbb{N}$, let $i \in \{0, 1, \ldots, d-1\}$, let $x_0, x_1 \ldots, x_{d-1} \in \mathbb{N}_0$, let $y \in \mathbb{Z}$, let $(R, M) \in \mathcal{M}^d$ such that*

$$d^j_M(-j) \leq x_j \quad \forall i \in \{0, 1, \ldots, d-1\}, \quad \operatorname{beg}(M) \geq y.$$

Then

$$\nu^i_M \geq -G^i_d(x_0, x_1, \ldots, x_{d-1}, y).$$

Proof. This is immediate by (4.10) and (3.8). □

We first aim to apply this result in she sheaf theoretic context using the concept of postulation number of sheaf as defined in (3.7)E). To this end we now introduce some apropriate notation.

5.2. **Notation.** Let $t \in \mathbb{N}_0$ and let $i \in \{0, 1, \ldots, t\}$. We then define the *bounding function*

$$L^i_t : \mathbb{N}^{t+1}_0 \to \mathbb{Z}$$

by the prescription

$$L^i_t(x_0, x_1, \ldots, x_t) := -G^{i+1}_{t+1}(x_0, x_1, \ldots, x_t, 0), \quad \forall x_0, x_1, \ldots, x_t \in \mathbb{N}_0,$$

where the function

$$G^{i+1}_{t+1} : \mathbb{N}^t_0 \times \mathbb{Z} \to \mathbb{Z}$$

is defined according to (4.6).

Now, we are ready to formulate and to prove our first main application of (5.1), which says that the *cohomology diagonal of a coherent sheaf $\mathcal{F}$ over a projective K-scheme X bounds the cohomological postulation numbers of the sheaf $\mathcal{F}$.*

5.3. **Theorem.** *Let $t \in \mathbb{N}_0$, let $i \in \{0, 1, \ldots, t\}$, let $x_0, x_1, \ldots, x_t \in \mathbb{N}_0$, and let $(X, \mathcal{F}) \in \mathcal{S}^t$ such that*

$$h^j_{\mathcal{F}}(-j) = h^i(X, \mathcal{F}(-j)) \leq x_j \quad \forall j \in \{0, 1, \ldots, t\}.$$

Then

$$\nu^i_{\mathcal{F}} \geq L^i_t(x_0, x_1, \ldots, x_t).$$

Proof. We chose a pair $(R, M) \in \mathcal{M}^{t+1}$, such that $(X, \mathcal{F}) = (\mathrm{Proj}(R), \widetilde{M})$. As $\widetilde{M} = \widetilde{M_{\geq 0}}$ we may replace M by $M_{\geq 0}$ and hence assume that

$$\mathrm{beg}(M) \geq 0.$$

Keep in mind that $\dim_R(M) = t + 1$ and that by (1.5)C) we have

$$d^j_M(-j) = h^j(X, \mathcal{F}(-j)) \leq x_j, \quad \forall j \in \{0, 1, \dots, t\}.$$

So, we may apply (5.1) with $y = 0$ and with $i + 1$ instead of i and obtain

$$\nu^{i+1}_M \geq -G^{i+1}_{t+1}(x_0, x_1, \dots, x_t, 0) = L^i_t(x_0, x_1 \dots, x_t).$$

By (3.7)E)a) we have in addition that $\nu^i_{\mathcal{F}} = \nu^{i+1}_M$ provided that $i > 0$. In these cases we therefore have our claim. So, it remains to consider the case $i = 0$. By (3.7)E)b) and the previous estimate we have

$$\nu^0_{\mathcal{F}} \geq \min\{\nu^1_M, 0\} \geq \min\{L^0_s(x_0, x_1, \dots, x_s), 0\}.$$

According to (4.6)(iii) we have

$$L^0_s(x_0, x_1, \dots, x_s) = -G^1_{s+1}(x_0, x_1, \dots, x_s, 0) < 0,$$

so that indeed $\nu^0_{\mathcal{F}} \geq L^0_s(x_0, x_1, \dots, x_s)$, as requested. $\square$

In order to draw conclusions from the previous estimate we need a further result, which was originally shown in [Br-Matt-Mi1]. In these lectures, we will not prove it. For a complete and self-contained proof we recommend to consider Section 8 of [Br8].

5.4. **Theorem.** *Let $t \in \mathbb{N}_0$ and let $(X, \mathcal{F}) \in \mathcal{S}^t$. Then*

a) $\mathrm{reg}(\mathcal{F}) \leq \left(2 \sum_{i=1}^t \binom{t-1}{i-1} h^i_{\mathcal{F}}(-i)\right)^{2^{t-1}} =: B.$

b) $\sum_{i=1}^t \binom{t-1}{i-1} h^i_{\mathcal{F}}(n - i) \leq \frac{B}{2}$ *for all* $n \in \mathbb{N}_0$.

Now, combining Theorems 5.3 and 5.4 and using the terminology introduced in Section 1, we get the following finiteness result:

5.5. **Theorem.** *Let $t \in N_0$, let $r \in \mathbb{Z}$ and let $x_0, x_1, \dots, x_t \in \mathbb{N}_0$. Then the class*

$$\mathcal{D} = \mathcal{D}_{x_0, x_1, \dots, x_t} := \{(X, \mathcal{F}) \in \mathcal{S}^t \mid h^i_{\mathcal{F}}(r - i) \leq x_i, \quad \forall i \leq t\}$$

is of finite cohomology.

Proof. After twisting we may assume that $r = 0$. Let B as in Theorem 5.4 a) and set

$$C := \min\{-t, \min\{-L^i_t(x_0, x_1, \dots, x_t) \mid i = 0, 1, \dots, t\}\}.$$

According to Theorem 5.4 A0,b) the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{S} := \{(i, n) \mid 1 \leq i \leq t, \quad n \geq -i\}.$$

Moreover, the inequality a) proved in the current of the proof of (4.5) together with the observations made in (1.5)C) yields that

a) $h^i_{\mathcal{F}}(n) \leq \sum_{j=0}^{i} \binom{-n-j-1}{i-j} \left[\sum_{k=j}^{i-j} \binom{i-j}{k-j} x_k \right]$ for all $i \in \mathbb{N}_0$, all $n \leq -i$ and all pairs $(X, \mathcal{F}) \in \mathcal{D}$.

This implies that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{T} := \{(i, n) \mid 0 \leq i \leq t, C - t - 2 \leq n \leq -i\}.$$

By Theorem 5.4 we have $\nu^i_{\mathcal{F}} \geq C$ and hence

b) $p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n)$ for all $(X, \mathcal{F}) \in \mathcal{D}$, for all $i \in \mathbb{N}_0$ and for all $n \leq C - 1$.

In particular, for all $(i, n) \in \mathbb{T}$ we have $p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n)$. As $\mathcal{D}$ is of finite cohomology on $\mathbb{T}$ and all polynomials $p^i_{\mathcal{F}}$ are of degree at most t, it follows:

c) The set $\{p^i_{\mathcal{F}} \mid 0 \leq i \leq t, (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

But now another use of the previously observed coincidence b) of cohomological Hilbert functions $h^i_{\mathcal{F}}$ and cohomological Serre polynomials $p^i_{\mathcal{F}}$ in the range $n \leq C - 1$ for all $(X, \mathcal{F}) \in \mathcal{D}$, it follows that the family $\mathcal{D}$ is also of finite cohomology on the set

$$\mathbb{U} := \{(i, n) \mid 0 \leq i \leq t, \quad n \leq -i\}.$$

It thus remains to show that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{V} := \{0\} \times \mathbb{N}.$$

Observe first, that by statement c) and by (3.7)D)d) the set of Serre-polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite. So, from (1.8)C)c) and Theorem 5.4 a) it follows that the set of functions

$$\{h^0_{\mathcal{F}} \upharpoonright_{\mathbb{Z}_{>B}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite. But this implies, that the class $\mathcal{D}$ is of finite cohomology on the set

$$\mathbb{W} := \{0\} \times \mathbb{Z}_{>B}.$$

It remains to be shown that the class $\mathcal{D}$ is of finite cohomology on the finite set

$$\mathbb{I} := \mathbb{V} \setminus \mathbb{W} = \{0\} \times \{1, 2, \ldots, B\}.$$

This means to show that the set of functions

$$\mathcal{H} := \{h^0_{\mathcal{F}} \upharpoonright_{\{1,2,\ldots,B\}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite. As in statement b) in the proof of (4.5) we see that $d^0_M(n-1) \leq d^0_M(n)$ for all $n \in \mathbb{Z}$ and all $(R, M) \in \mathcal{M}^{t+1}$. So, by (1.5)C) we get

$$h^0_{\mathcal{F}}(n-1) \leq h^0_{\mathcal{F}}(n), \quad \forall n \in \mathbb{Z}, \quad \forall (X, \mathcal{F}) \in \mathcal{S}^t.$$

As the class $\mathcal{D}$ is of finite cohomology on the singleton set $\{(0, B+1)\}$ it follows immediately, that the set $\mathcal{H}$ is finte. $\square$

Rephrasing the previous result we may say:

5.6. **Corollary.** *Let $t \in \mathbb{N}_0$ and let $r \in \mathbb{Z}$. Then, a subclass $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if it is of finite cohomology on the diagonal subset*

$$\Delta = \Delta_r^t = \Delta_r^t := \{(i.r - i) \mid i = 0, 1, \ldots, t\}.$$

Proof. This is immediate by Theorem 5.5. □

5.7. **Definition and Remark.** A) *(Bounding Sets for Cohomology)* Let $t \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^t$. A subset

$$\mathbb{B} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$$

is called a *bounding set for cohomology with respect to the class $\mathcal{D}$*, if each subclass $\mathcal{E} \subseteq \mathcal{D}$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all. If $\mathbb{B}$ is a bounding set with respect to the full class $\mathcal{S}^t$ it is called a *bounding set for cohomology (at all).*

B) *(Rephrasing Corollary 5.6)* Keep the notations and hypotheses of part A). We now may rephrase Corollary 5.6 as follows:

a) Each set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which contains a diagonal subset $\Delta = \Delta_r^t$ is a bounding set for cohomology.

The reformulation of Corollary 5.6 suggested above gives rise to the question, whether there is a combinatorial charcterisation of all bounding sets. In order to deal with this problem, we define the notion of *Quasi-Diagonal Set.*

5.8. **Definition.** *(Quasi Diagonal Sets)* Let $t \in \mathbb{N}_0$. A subset

$$\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$$

is said to be a *quasi-diagonal (subset)* if there are integers $n_t < n_{t-1} < \ldots < n_0$ such that

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\}.$$

Observe, that each diagonal subset $\Delta = \Delta_r^t = \{(i, r - i) \mid i = 0, 1 \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is a quasi-diagonal subset.

We next prove the following auxiliary result.

5.9. **Lemma.** *Let $t \in \mathbb{N}_0$, let $n_t < n_{t-1} < \ldots < n_0$ be integers and let $\mathcal{D} \subseteq \mathcal{S}^t$ be a subclass which is of finite cohomology on the quasi-diagonal subset*

$$\Sigma := \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}.$$

Then, the class $\mathcal{D}$ is of finite cohomology on the diagonal set

$$\Delta = \Delta_{t+n_t}^t := \{(i, t + n_t - i) \mid i = 0, 1, \ldots, t\}.$$

Proof. . After twisting, we may assume that $n_t = -t$, so that

$$\Delta = \Delta_0^t = \{(i, -i) \mid i = 0, 1, \ldots, t\}.$$

Once more we use the estimate a) from the proof of 4.5 and translate it to the sheaf theoretic context by means of the observation made in (1.5)C). What we get is the estimate

a) $h^i_{\mathcal{F}}(n) \leq \sum_{j=0}^{i} \binom{-n-j-1}{i-j} \big[\sum_{l=0}^{i-j} \binom{i-j}{l} h^{i-1}_{\mathcal{F}}(l-i) \big], \forall i \in \mathbb{N}_0, \forall n \leq -i, \forall (X, \mathcal{F}) \in \mathcal{S}^t$.

Now, we prove our claim by induction on the number

$$\sigma = \sigma(\Sigma) := n_0 - t \quad (\geq t).$$

If $\sigma = t$ we have $\Sigma = \Delta$ and our claim is obvious.

So, let $\sigma > t$. Then there is some $i \in \{0, 1, \ldots, t-1\}$ such that $n_i - n_{i+1} > 1$. We chose $i := i(\Sigma)$ minimal with this property and proceed by induction in $i = i(\Sigma)$. Assume first, that $i = 0$. Then $n_1 + 1 < n_0$ and it follows by the above statement A) applied with $i = 0$ that

$$h^0_{\mathcal{F}}(n_1 + 1) = h^0_{\mathcal{F}(n_0)}(n_1 + 1 - n_0) \leq h^0_{\mathcal{F}(n_0)}(0) = h^0_{\mathcal{F}}(n_0), \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

But this implies that the class $\mathcal{D}$ is of finite cohomology on the set

$$\Sigma' := \{(0, n_1 + 1)\} \cup \{(j, n_j) \mid j = 1, 2 \ldots, t\}.$$

But for this set we also have $\sigma(\Sigma') < \sigma(\Sigma) = \sigma$. Therefore, by induction the class $\mathcal{D}$ is of finite cohomology on the set Δ.

Now, let $i > 0$. Then clearly $n_i - 1 - n_0 = -i - 1$ and $\mathcal{D}$ is of finite cohomology on the non-empty set

$$\{(i - l, n_0 + l - i) \mid l = 0, 1, \ldots, i\} = \{(k, n_k) \mid k = 0, 1, \ldots, i\} \subseteq \Sigma.$$

So, there is some $h \in \mathbb{N}_0$ such that

$$h^{i-l}_{\mathcal{F}(n_0)}(l - i) = h^{i-l}_{\mathcal{F}}(n_0 + l - i) \leq h, \quad \forall l \in \{0, 1, \ldots, i\}, \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

By the above statement a) it follows that there is some $h' \in \mathbb{N}_0$ such that

$$h^i_{\mathcal{F}}(n_i - 1) = h^i_{\mathcal{F}(n_0)}(n_i - 1 - n_0) = h^i_{\mathcal{F}(n_0)}(-i - 1) \leq h', \quad \forall (X, \mathcal{F}) \in \mathcal{D}.$$

From this we obtain that the class $\mathcal{D}$ is of finite cohomology on the set

$$\Sigma'' := \{(j, n_j) \mid j = 0, 1, \ldots, i-1\} \cup \{(i, n_i - 1)\} \cup \{(k, n_k) \mid k = i+1, i+2, \ldots, t\}.$$

As $i(\Sigma'') = i(\Sigma) - 1 = i - 1$, we may conclude by induction. $\square$

Now, we can deduce the following result.

5.10. **Proposition.** *Let $t \in \mathbb{N}_0$ let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ be a subset which contains a quasi-diagonal subset*

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\}, \quad n_t < n_{t-1} < \ldots < n_0$$

and let $\mathcal{D} \subseteq \mathcal{S}^t$ a subclass which is of finite cohomology on $\mathbb{S}$. Then, the class $\mathcal{D}$ is of finite cohomology at all.

Proof. This is clear by Corollary 5.6 and Lemma 5.9. $\square$

In particular, we can say.

5.11. **Corollary.** *Let $t \in N_0$. Then each set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ which contains a quasi-diagonal subset Σ is a bounding set for cohomology.*

Proof. This is immediate by Proposition 5.10. □

So, we have seen, that for a set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ a sufficient condition for being a bounding set for cohomology is to contain a quasi-diagonal. It is natural to ask whether this contition is also necessary. This is indeed the case, as stated by the following result.

5.12. **Theorem.** *Let $t \in \mathbb{N}_0$. Then, a set $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ is a bounding set for cohomology, if and only if it contains a quasi-diagonal set.*

Proof. The sufficiency of the condition to contain a quasi-diagonal is stated in Corollary 5.11. The necessity needs some extra work: one supposes that the set $\mathbb{S}$ does not contain a quasi-diagonal. Then, using approriate vector bundles on certain Segre products one constructs families of pairs $(X, \mathcal{F}) \in \mathcal{S}^t$ which are of finite cohomology on $\mathbb{S}$ but not on the set $\{0, 1, \ldots, t\} \times \mathbb{Z}$. For a detailed proof see [Br-Ja-Li2] (4.6). □

We now give a number of applications of the previous results, which generalize what we said in (1.9) about Hilbert schemes. We do this in the form of exercise. We begin with linking classes of finite cohomology to classes of bounded regularity. If you feel some lack of familiarity with sheaf theory, we recommend to translate all occuring statements to the module-theoretic setting and to prove the requested claims in this form.

5.13. **Exercise and Remark.** A) *(Specifying classes of Finite Cohomology)* Let $t \in \mathbb{N}_0$ and let $\mathcal{D} \subseteq \mathcal{S}^t$ be a subclass. Fix a quasi-diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}, \quad n_t < n_{t-1} < \ldots < n_0.$$

Then Proposition 5.10 says that the class $\mathcal{D}$ is of finite cohomology if and only if the set

$$\{h^i(X, \mathcal{F}(n_i)) \mid (X, \mathcal{F}) \in \mathcal{D}\} = \{h^i_{\mathcal{F}}(n_i) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

is finite for all $i \in \{0, 1, \ldots, t\}$. So, the $t + 1$ numerical invariants $h^i_{\mathcal{F}}(n_i)$ with $i = 0, 1, \ldots, t$ may be used to specify subclasses $\mathcal{D} \subseteq \mathcal{S}^t$ of finite cohomology. Indeed, specifying such classes by subjecting numerical invariants to some conditions, is a basic issue. In this spirit we suggest to prove the following statement as an exercise.

a) The class $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if there are integers $r \in \mathbb{Z}$ and $h \in \mathbb{N}_0$ such that $\operatorname{reg}(\mathcal{F}) \leq r$ and $h^0(X, \mathcal{F}(r)) \leq h$ for all pairs $(X, \mathcal{F}) \in \mathcal{D}$.

B) *(Classes of Bounded Regularity)* We say that the class $\mathcal{D} \subseteq \mathcal{S}^t$ is *of bounded regularity* if the set of integers

$$\{\operatorname{reg}(\mathcal{F}) \mid (X, \mathcal{F}) \in \mathcal{D}\}$$

has an upper bound in $\{-\infty\} \cup \mathbb{Z}$. Prove the following statement.

b) The class $\mathcal{D} \subseteq \mathcal{S}^t$ is of finite cohomology if and only if it is of bounded regularity and the set of Serre polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

C) *(Regularity and Classes of Subsheaves and Quotient Sheaves)* Let $t \in \mathbb{N}_0$. we consider the class

$$\mathcal{S}^{\leq t} := \bigcup_{i=0}^{t} \mathcal{S}^i$$

of all pairs $(X, \mathcal{F})$ in which X is a projective scheme over some field K and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules with $\dim(\mathcal{F}) \leq t$. The notions of *subclass $\mathcal{D} \subseteq \mathcal{S}^{\leq t}$ of finite cohomology* and of subclass *of bounded regularity* are defined in the obvious way as previousely. Now, let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq t}$. We say that $\mathcal{D}$ is *a class of subsheaves with respect to* $\mathcal{C}$ if for all pairs $(X, \mathcal{F}) \in \mathcal{D}$ there is a monomorphism of sheaves $0 \to \mathcal{F} \xrightarrow{h} \mathcal{G}$ with $(X, \mathcal{G}) \in \mathcal{C}$. Prove the following statement

a) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq s}$ be such that $\mathcal{C}$ is of finite cohomology and $\mathcal{D}$ is a class of subsheaves with respect to $\mathcal{C}$. Then the class $\mathcal{D}$ is of finite cohomology if and only if it is of bounded regularity.

If X is a projective scheme over some field K and $\mathcal{F}, \mathcal{G}$ are two coherent sheaves of $\mathcal{O}_X$-modules we say that $\mathcal{F}$ is *a quotient of* $\mathcal{G}$ if there is an epimorphism of sheaves $\mathcal{G} \xrightarrow{h} \mathcal{F} \to 0$. Accordingly we say that $\mathcal{D}$ is *a class of quotient sheaves with respect to* $\mathcal{C}$ if for each pair $(X, \mathcal{F}) \in \mathcal{D}$ there is a pair $(X, \mathcal{G}) \in \mathcal{C}$ such that $\mathcal{F}$ is a quotient of $\mathcal{G}$. Prove the following statement.

b) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^{\leq s}$ be such that $\mathcal{C}$ is of finite cohomology and $\mathcal{D}$ is a class of quotient sheaves with respect to $\mathcal{C}$. Then the class $\mathcal{D}$ is of finite cohomology if and only if it is of bounded regularity.

D) *(Serre Polynomials and Classes of Subsheaves and Quotient Sheaves)* We now generalize what was said about Hilbert schemes in Example 1.9. Keep the notations and hypotheses of part C). Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}^t$ be suclasses. We recall the following fact, which generalizes Mumford's basic Bounding Result guven in [Mu1] (see [Br8]).

a) For each polynomial $p \in \mathbb{Q}[X]$ of degree t and each integer $\rho \in \mathbb{Z}$ there is a number $\beta = \beta_r(\rho)$ such that for each projective scheme X and each epimorphism of coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{H} \to 0$ with $\operatorname{reg}(\mathcal{F}) \leq \rho$ and $P_{\mathcal{H}} = p$ one has $\operatorname{reg}(\mathcal{H}) \leq \beta$.

Use this, to prove the following statement

b) Let $\mathcal{D}$ be a class of subsheaves (resp. of quotient sheaves) with respect to $\mathcal{C}$ and assume that $\mathcal{C}$ is of finite cohomology. Then the class $\mathcal{D}$ is of finite cohomology if and only if the set of Serre polynomials $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite

The following special case of the previous statement covers most closely our observation on Hilbert schemes made in Example 1.9. Fix a pair $(X, \mathcal{G}) \in \mathcal{S}^{\leq t}$ and let $\mathcal{D}$ be a class of subsheaves or of quotient sheaves of $\mathcal{G}$. Show that the following statements are equivalent.

(i) $\mathcal{D}$ is a class of finite cohomology.
(ii) $\mathcal{D}$ is a class of bounded regularity.
(iii) The set $\{P_{\mathcal{F}} \mid (X, \mathcal{F}) \in \mathcal{D}\}$ is finite.

Now we give another remark, which concerns bounding sets for cohomology with respect to specific subclasses of $\mathcal{S}^t$.

5.14. **Remark.** A) *(Bounding Sets for Classes of Vector Bundles)* Let $T \in \mathbb{N}$. It is natural to ask, whether for appropriate subclasses of $\mathcal{D} \subseteq \mathcal{S}^t$ there are more bounding sets for cohomology than those specified by Theorem 5.12 . A particularly intersting setting for this question is given as follows: Let K be a field and let

$$\mathcal{V} \subseteq \mathcal{S}^t$$

be the family of all algebraic vector bundles over the projective space $\mathbb{P}^t_K = \mathrm{Proj}(K[X_0, X_1, \ldots, X_t])$ and let $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$. We say that $\mathbb{S}$ *bounds cohomology of vector bundles (over* $\mathbb{P}^t_K$*)*, if each subclass $\mathcal{D} \subseteq \mathcal{V}$ which is of finite cohomology on $\mathbb{S}$ is of finite cohomology at all. We do not know a precise combinatorial characterization of those subsets $\mathbb{S}$ which bound cohomology of vector bundles. What is shown in the Master thesis [Ke] is the following special result

a) If the sets $\mathbb{S} \cap (\{t\} \times \mathbb{Z}_{<0})$ and $\mathbb{S} \cap (\{0\} \times \mathbb{Z}_{>0})$ are both finite, then the set $\mathbb{S}$ bounds cohomology of vector bundles if and only if it contains a quasi-diagonal subset of $\{0, 1, \ldots, t\} \times \mathbb{Z}$.

Clearly this means in particular;

b) A finite subset $\mathbb{S} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ bounds cohomology of vector bundles if and only if it contains a quasi-diagonal subset.

So, here is a problem;

c) Is there a (necessarily infinite) set $\mathbb{S} \subseteq \{0, 1, \ldots, s\} \times \mathbb{Z}$ which contains no quasi-diagonal subset $\Sigma \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}$ but which does bound cohomology of vector bundles?

B) *(Counting Cohomology Tables)* Keep the notations of part A). Fix an arbitrary (quasi-)diagonal subset

$$\Sigma = \{(i, n_i) \mid i = 0, 1, \ldots, t\} \subseteq \{0, 1, \ldots, t\} \times \mathbb{Z}, \quad (n_t < n_{t-1} < \ldots < n_0).$$

and fix a family of non-negative integers

$$\bar{h} := (h^i)_{i=0}^t.$$

Then clearly we know by Theorem 5.12, that the number of cohomology tables

$$N_{\Sigma,\bar{h}} := \#\{h_{\mathcal{F}} \mid (X,\mathcal{F}) \in \mathcal{S}^t : h^i(X,\mathcal{F}(n_i)) = h_i, \quad i = 0,1\ldots,t\}$$

is finite. Going tedeously through our arguments on could indeed get some upper bound for this number, at least in the case where Σ is the standard diagonal subset $\{(i,-i) \mid i = 0,1,\ldots,t\}$. So, one could get stuck to the idea of counting all possible cohomology tables with a given standard cohomology diagonal, or at least to bound there number in a satisfactory way. Clearly, one cannot expect, that a bound which is obtained on use of the arguments of our proves will be satisfactory. The enourmous discrepancy between the expected and the actual number of cohomology tables is made evident in the Master thesis [Cat].

So, our bounding results are not appropriate to perform quantitative arguments in the sense of counting cohomology tables. On the other hand our results furnish at least the equivalence of the following statements, which also follows from the properties of cohomological patterns (see (2.5)) - and whose proof we suggest as an exercise.

(i) $\mathcal{F} = 0$.
(ii) $h^i(X,\mathcal{F}(-i)) = 0$ for all $i \in \{0,1,\ldots,s\}$.
(iii) There is some $t \in \mathbb{Z}$ such that $H^i(X,\mathcal{F}(t-i)) = 0$ for all $i \in \{0,1,\ldots,s\}$.
(iv) $h^i = 0$ for all $i \in \{0,1,\ldots,s\}$.

C) *(Characterizing Cohomology Tables)* Refining what we presented in Section 2 and pushing further the idea of counting cohomology tables, one could try to characterise all families $\left(h^i_n\right)_{(i,n)\in\{0,1,\ldots,t\}\times\mathbb{Z}}$ of h^i_n which occur as cohomology table $h_{\mathcal{F}}$ of some pair $(X,\mathcal{F}) \in \mathcal{S}^t$. We do not know the answer to this problem. Clearly it would be much more interesting and more challanging to answer this question for some specific classes $\mathcal{D} \in \mathcal{S}^t$. So, one could think to choose $\mathcal{D}$ to be the class of all pairs $(X,\mathcal{O}_X)$, where $X \subseteq \mathbb{P}^r_K$ runs through all closed subschemes with a given Serre polynomial $P_{\mathcal{O}_X} = p$ with $\deg(p) = t$, hence through the class of closed subschemes parametrized by $\mathrm{Hilb}^p_{\mathbb{P}^r}$. Another challange would be to attack this problem in the case where $\mathcal{D}$ is the class of all vector bundles $\mathcal{E}$ of given rank $\mathrm{rank}(\mathcal{E}) = r$ over a fixed projective space $\mathbb{P}^t_K$ or with given Serre polynomial $P_{\mathcal{E}}$.

6. Bibliographical Hints

We append to these notes the full bibliography of [Br8]. We now shall give a rather rough classification of the occuring references, in the hope to motivate interested readers to penetrate further into the subject or to clarify the background which we considered as known in our lectures.

1. General Commutative Algebra, Homological Algebra and Algebraic Geometry:

[Br0], [Br-Bo-Ro], [Bru-Her], [E1], [Ev-Gri], [Gro-D], [Gro5], [H1], [Kun1], [Mat], [R], [Sh].

2. General Local Cohomology and Sheaf Cohomology:

[Br-Fu-Ro], [Br-Sh1], [Gro-D], [Gro2], [H1], [Se].

3. Structure, Vanishing and Bounding Results for Local Cohomology and Sheaf Cohomology:

[A-Br], [B-Mu], [Br1], [Br2], [Br3], [Br4], [Br8], [Br-He], [Br-Ja-Li1], [Br-Ja-Li2], [Br-K-Sh], [**?**], [Br-Matt-Mi1], [Br-Matt-Mi2], [Br-N], [Br-Sh1], [Br-Sh2], [Br-Sh3], [Cat], [En], [Fa1], [Fa2], [Fu], [Gro4], [H1], [H2], [K], [Ke], [Ko], [L], [M], [Matt], [Mi-N-P], [Mu2], [Ro], [Se], [Sev], [Si], [Tru], [Z].

4. Castelnuovo-Mumford regularity and its Historic Background:

[B-Mu], [B-St], [Bäc], [Be1],, [Be2], [Br2], [Br4], [Br5], [Br8], [Br-Gö], [Br-Ja-Li1], [Br-Matt-Mi1], [Br-Sh1], [Br-Vo], [Bu], [C], [Cav], [Cav-Sb], [Ch-F-N], [E-G], [G], [Gi], [Go1], [Gru-La-P], [Hen-Noe], [Herm], [Hi1], [Hi2], [La], [Mas-W], [Ma-Me], [Mu1], [O], [Pi], [Sei].

5. Hilbert Schemes

[Fu], [Go1], [Go2], [Gro6], [H1], [H3], [Mat], [P], [Pe-St].

5. Vector Bundles and their Cohomology

[A-Br], [Br4], [Cat], [El-Fo], [En], [Ev-Gri], [Gr-Ri], [Gro0], [Gro4], [H1], [Ho], [Ke], [Ko], [Matt], [Mu1], [Mu2], [Se], [Sev], [Z].

6. Cohomology Tables, Cohomological Patterns, Tameness and Asymptotic Behaviour of Cohomology

[Bä-Br], [Br4], [Br6], [Br7], [Br-Fu-Lim], [Br-Fu-T], [Br-He], [Br-K-Sh], [Br-Ku-Ro], [Br-Ro], [Br-Ro-Sa], [Cat], [Ch-Cu-Her-Sr], [K], [Ke], [Lim1], [Lim2], [Lim3], [Mat], [M], [Mi-N-P], [Rot-Seg], [Si].

7. Deficiency and Canonocal Modules [Br8], [Br-Sh1], [Bru-Her], [Her-Kun], [Sc1], [Sc2].

8. Related Work on Projective Varieties [A-Br], [Be1], [Be2], [Br1], [Br4], [Br-Sc1], [**?**], [Br-Sc3], [Br-Vo], [C], [En], [Gru-La-P], [H1], [Ko], [La], [Mat], [Mi-N-P], [Mu2], [Pi], [Se], [Sev], [Z].

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