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INTRODUCTION TO SHEAVES

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Introduced in the 1940s by Leray and Cartan, sheaves are a powerful tool for relating local and global phenomena on a space. Smooth differential forms of a given degree define a sheaf on a manifold. On a complex manifold, in addition to smooth differential forms, there are holomorphic differential forms that also define sheaves on the manifold. Cohomology with coefficients in the sheaf Ω^p of holomorphic p-forms gives invariants of the complex structure. Sheaves have become absolutely essential in modern algebraic geometry as well as certain areas of topology and complex analysis.

1. Presheaves

The functor $\mathcal{A}^*$ that assigns to every open set U on a manifold the vector space of C^∞ forms on U is an example of a presheaf. By definition a ***presheaf*** $\mathcal{F}$ of abelian groups on a topological space X is a function that assigns to every open set U in X an abelian group $\mathcal{F}(U)$ and to every inclusion of open sets $i_U^V\colon V \to U$ a group homomorphism $\mathcal{F}(i_U^V) := \rho_V^U$, called the ***restriction*** from U to V,

$$\rho_V^U\colon \mathcal{F}(U) \to \mathcal{F}(V),$$

such that the system of restrictions ρ_V^U satisfies the following properties:

(i) (identity) $\rho_U^U = \mathbb{1}_{F(U)}$, the identity map on $\mathcal{F}(U)$;
(ii) (transitivity) if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$.

We refer to elements of $\mathcal{F}(U)$ as ***sections*** of $\mathcal{F}$ over U. The group $\mathcal{F}(U)$ is also written $\Gamma(U,\mathcal{F})$. Elements of $\Gamma(X,\mathcal{F})$ are called ***global sections*** of $\mathcal{F}$.

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a ***morphism*** $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves is a collection of group homomorphisms $f_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set U in X, that commute with the restrictions, i.e., such that each diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho_V^U"'] & \mathcal{G}(U) \arrow[d, "\rho_V^U"] \\ \mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V) \end{tikzcd} \tag{1.1}$$

is commutative. Although we write both vertical maps as ρ_V^U, they are in fact not the same map; the first one is $\mathcal{F}(i_U^V)$ and the second is $\mathcal{G}(i_U^V)$. If we write $\omega|_U$ for $\rho_V^U(\omega)$, then the diagram (1.1) is equivalent to $f_V(\omega|_V) = f_U(\omega)|_V$ for all $\omega \in \mathcal{F}(U)$. In practice, we often omit the subscripts in f_U and f_V and write them simply as f.

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For any topological space X, let $\mathrm{Open}(X)$ be the category in which the objects are open subsets of X and for any two open subsets U, V of X, the set of morphisms from V to U is

$$\mathrm{Hom}(V,U) := \begin{cases} \{\text{inclusion } i_U^V \colon V \to U\} & \text{if } V \subset U, \\ \text{the empty set } \varnothing & \text{otherwise.} \end{cases}$$

In functorial language, a presheaf of abelian groups is simply a contravariant functor from the category $\mathrm{Open}(X)$ to the category of abelian groups, and a morphism of presheaves is a natural transformation from the functor $\mathcal{F}$ to the functor $\mathcal{G}$. What we have defined are presheaves of abelian groups; it is possible to define similarly presheaves of vector spaces, of algebras, and indeed of objects in any category, but all the presheaves that we consider will be presheaves of abelian groups.

Example 1.1. The ***zero presheaf*** $\mathcal{F}$ on a topological space X associates to every open set U the zero group $\mathcal{F}(U) = 0$ and to every inclusion $V \subset U$ the zero map $\mathcal{F}(U) \to \mathcal{F}(V)$.

Example. If G is an abelian group, we define the ***presheaf of locally constant G-valued functions on X*** (constant on connected components) to be the presheaf $\underline{G}$ that associates to every open set U in X the group

$$\underline{G}(U) := \{\text{locally constant functions } f \colon U \to G\}$$

and to every inclusion of open sets $V \subset U$ the restriction $\rho_V^U \colon \underline{G}(U) \to \underline{G}(V)$ of locally constant functions.

2. The Stalk of a Presheaf

On a smooth manifold M, the function that assigns to every open set $U \subset M$ the group $C^\infty(U)$ of C^∞ real-valued functions on U is a presheaf. As we know from manifold theory, the behavior of C^∞ functions at a point is encoded in the the *germs* of the functions at the point. The corresponding notion for a presheaf is the *stalk* of the presheaf at a point. To define the stalk, we recall an algebraic construction called the *direct limit* of a direct system of groups.

A ***directed set*** is a set I with a binary relation $\leq$ satisfying

(i) (reflexivity) for all $a \in I$, $a \leq a$,
(ii) (transitivity) for all $a, b, c \in I$, if $a \leq b$ and $b \leq c$, then $a \leq c$,
(iii) (upper bound) for all $a, b \in I$, there is an element $c \in I$, called an ***upper bound*** of a and b, such that $a \leq c$ and $b \leq c$.

We often write $b \geq a$ if $a \leq b$.

A ***direct system of groups*** is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set I and a collection of group homomorphisms $f_b^a \colon G_a \to G_b$ indexed by pairs $a \leq b$ in I such that

(i) $f_a^a = \mathbb{1}_{G_a}$, the identity map on G_a,
(ii) $f_c^a = f_c^b \circ f_b^a$ for $a \leq b \leq c$ in I.

On the disjoint union $\coprod_i G_i$ we introduce an equivalence relation $\sim$ by decreeing two elements g_a in G_a and g_b in G_b to be equivalent if there exists an upper bound c of a and b such that $f_c^a(g_a) = f_c^b(g_b)$ in G_c. The ***direct limit*** of the direct system, denoted by $\varinjlim_{i \in I} G_i$, is the quotient of the disjoint union $\coprod_i G_i$ by the equivalence relation $\sim$; in other words, two elements of $\coprod_i G_i$ represent the same element in the direct limit if they are "eventually equal." We make the direct limit $\varinjlim G_i$ into a group by defining $[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)]$, where c is an upper bound of a and b and $[g_a]$ is the equivalence class of g_a. It is easy to check that the addition $+$ is well defined and that with this operation the direct limit $\varinjlim G_i$ becomes a group; moreover, if all the groups G_i

are abelian, then so is their direct limit. Instead of groups, one can obviously also consider direct systems of modules, rings, algebras, and so on.

Example. Fix a point p in a manifold M and let I be the directed set consisting of all neighborhoods of p in M, with $\leq$ being reverse inclusion: $U \leq V$ if and only if $V \subset U$. Let $C^\infty(U)$ be the ring of C^∞ functions on U. Then $\{C^\infty(U)\}_{U \ni p}$ is a direct system of rings and its direct limit $C_p^\infty := \varinjlim_{U \ni p} C^\infty(U)$ is precisely the ring of germs of C^∞ functions at p.

If $\mathcal{F}$ is a presheaf of abelian groups on a topological space X and p is a point in X, then $\{\mathcal{F}(U)\}_{U \ni p}$, where U ranges over all open neighborhoods of p, is a direct system of abelian groups. The direct limit $\mathcal{F}_p := \varinjlim_{U \ni p} \mathcal{F}(U)$ is called the ***stalk*** of $\mathcal{F}$ at p. An element of the stalk $\mathcal{F}_p$ is called a ***germ*** of sections at p. For example, the ring C_p^∞ is the stalk at p of the presheaf $C^\infty(\ )$ of C^∞ functions on the manifold M.

A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ over a topological space X induces a morphism of stalks $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ at each $p \in X$ by sending the germ at p of a section $s \in \mathcal{F}(U)$ to the germ at p of the section $\varphi(s) \in \mathcal{G}(U)$. The morphism $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ of stalks is also called the ***stalk map*** at p.

3. Sheaves

The stalk of a presheaf at a point embodies in it the local character of the presheaf about the point. However, in general there is no relation between the global sections and the stalks of a presheaf.

Example 3.1. If G is an abelian group and $\mathcal{F}$ is the presheaf on a topological space X defined by $\mathcal{F}(X) = G$ and $\mathcal{F}(U) = 0$ for all $U \neq X$, then all the stalks $\mathcal{F}_p$ vanish, but $\mathcal{F}$ is not the zero presheaf.

A *sheaf* is a presheaf with two additional properties that link the global and local sections of the presheaf. In practice, most of the presheaves one encounters are sheaves. Unlike the example in the preceding paragraph, a sheaf all of whose stalks vanish has no nonzero global sections (Example 5.2).

Definition 3.2. A ***sheaf*** $\mathcal{F}$ of abelian groups on a topological space X is a presheaf satisfying two additional conditions for any open set $U \subset X$ and any open cover $\{U_i\}$ of U:

(i) (uniqueness axiom) if $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$;

(ii) (gluing axiom) if $\{s_i \in \mathcal{F}(U_i)\}$ is a collection of sections such that
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j,$$
then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.

Suppose there is an ordering on the index set I of the open cover $\{U_i\}_{i \in I}$, and consider the sequence of maps
$$0 \to \mathcal{F}(U) \xrightarrow{r} \prod_i \mathcal{F}(U_i) \xrightarrow{\delta} \prod_{i<j} \mathcal{F}(U_i \cap U_j), \tag{3.1}$$
where r is the restriction $r(\omega)_i = \omega|_{U_i}$ and δ is the Čech coboundary operator
$$(\delta\omega)_{ij} := \omega_j|_{U_{ij}} - \omega_i|_{U_{ij}}.$$
Then the two sheaf axioms (i) and (ii) are equivalent to the exactness of the sequence (3.1) at $\mathcal{F}(U)$ and at $\prod_i \mathcal{F}(U_i)$, respectively; i.e., the map r is injective and $\ker\delta = \operatorname{im} r$.

Example. For any open subset U of a topological space X, let $\mathcal{F}(U)$ be the abelian group of constant real-valued functions on U. If $V \subset U$, let $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ be the restriction of functions. Then $\mathcal{F}$ is a presheaf on X. Suppose X has nonempty disjoint subsets (for example, $X = \mathbb{R}^n$ with the standard topology). Then the presheaf $\mathcal{F}$ satisfies the uniqueness axiom but not the gluing axiom of a sheaf: if U_1 and U_2 are disjoint open sets in X, and $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ have different values, then there is no constant function s on $U_1 \cup U_2$ that restricts to s_1 on U_1 and to s_2 on U_2.

Example. Let $\underline{\mathbb{R}}$ be the presheaf on a topological space X that associates to every open set $U \subset X$ the abelian group $\underline{\mathbb{R}}(U)$ consisting of all *locally* constant real-valued functions on U. Then $\underline{\mathbb{R}}$ is a sheaf. More generally, if G is an abelian group, then the presheaf $\underline{G}$ of locally constant functions with values in G is a sheaf, called the ***constant sheaf*** with values in G.

Example. The zero presheaf 0 on a topological space in Example 1.1 is a sheaf.

Example. The presheaf $\mathcal{A}^k$ on a manifold that assigns to each open set U the abelian group of C^∞ k-forms on U is a sheaf.

Example. The presheaf $\mathcal{Z}^k$ on a manifold that associates to each open set U the abelian group of closed C^∞ k-forms on U is a sheaf.

4. The Sheaf Associated to a Presheaf

Associated to a presheaf $\mathcal{F}$ on a topological space X is another topological space $E_{\mathcal{F}}$, called the ***étalé space*** of $\mathcal{F}$. As a set, the étalé space $E_{\mathcal{F}}$ is the disjoint union $\coprod_{p \in X} \mathcal{F}_p$ of all the stalks of $\mathcal{F}$. There is a natural projection map $\pi\colon E_{\mathcal{F}} \to X$ that maps $\mathcal{F}_p$ to p. A ***section*** of the étalé space $\pi\colon E_{\mathcal{F}} \to X$ over $U \subset X$ is a map $s\colon U \to E_{\mathcal{F}}$ such that $\pi \circ s = \mathbb{1}_U$, the identity map on U. For any open set $U \subset X$, element $s \in \mathcal{F}(U)$, and point $p \in U$, let $s_p \in \mathcal{F}_p$ be the germ of s at p. Then the element $s \in \mathcal{F}(U)$ defines a section $\tilde{s}$ of the étalé space over U,

$$\begin{aligned} \tilde{s}\colon U &\to E_{\mathcal{F}}, \\ p &\mapsto s_p \in \mathcal{F}_p. \end{aligned}$$

The collection

$$\{\tilde{s}(U) \mid U \text{ open in } X,\ s \in \mathcal{F}(U)\}$$

of subsets of $E_{\mathcal{F}}$ satisfies the conditions to be a basis for a topology on $E_{\mathcal{F}}$. With this topology, the étalé space $E_{\mathcal{F}}$ becomes a topological space. By construction, the topological space $E_{\mathcal{F}}$ is locally homeomorphic to X. For any element $s \in \mathcal{F}(U)$, the function $\tilde{s}\colon U \to E_{\mathcal{F}}$ is a continuous section of $E_{\mathcal{F}}$. A section t of the étalé space $E_{\mathcal{F}}$ is continuous if and only if every point $p \in X$ has a neighborhood U such that $t = \tilde{s}$ on U for some $s \in \mathcal{F}(U)$.

Let $\mathcal{F}^+$ be the presheaf that associates to each open subset $U \subset X$ the abelian group

$$\mathcal{F}^+(U) := \{\text{continuous sections } t\colon U \to E_{\mathcal{F}}\}.$$

Under pointwise addition the presheaf $\mathcal{F}^+$ is easily seen to be a sheaf, called the ***sheafification*** or the ***associated sheaf*** of the presheaf $\mathcal{F}$. There is an obvious presheaf morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ that sends a section $s \in \mathcal{F}(U)$ to the section $\tilde{s} \in \mathcal{F}^+(U)$.

Example. For an open set U in a topological space X, let $\mathcal{F}(U)$ be the group of all *constant* real-valued functions on U. At each point $p \in X$, the stalk $\mathcal{F}_p$ is $\mathbb{R}$. The étalé space $E_{\mathcal{F}}$ is $X \times \mathbb{R}$, but not with its usual topology. A set in $E_{\mathcal{F}}$ is open if and only if it is of the form $U \times \{r\}$ for an open set $U \subset X$ and a number $r \in \mathbb{R}$. Thus, the topology on $E_{\mathcal{F}} = X \times \mathbb{R}$ is the product of the given topology on X and the discrete topology on $\mathbb{R}$. The sheafification $\mathcal{F}^+$ is the sheaf $\underline{\mathbb{R}}$ of *locally constant* real-valued functions.

EXERCISE 4.1 Prove that if $\mathcal{F}$ is a sheaf, then $\mathcal{F} = \mathcal{F}^+$. (*Hint*: Show that every continuous section $t\colon U \to E_{\mathcal{F}}$ is $\tilde{s}$ for some $s \in \mathcal{F}(U)$.)

Proposition 4.2. *For every sheaf $\mathcal{G}$ and every presheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$, there is a unique sheaf morphism $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that the diagram*

$$\begin{tikzcd}
\mathcal{F}^+ \arrow[dr, dashed, "\varphi^+"] & \\
\mathcal{F} \arrow[u, "\theta"] \arrow[r, "\varphi"'] & \mathcal{G}
\end{tikzcd} \tag{4.1}$$

commutes.

PROOF. The proof is straightforward and is left as an exercise. □

5. SHEAF MORPHISMS

Recall that all our sheaves are sheaves of abelian groups. A ***morphism*** $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves, also called a ***sheaf map***, is by definition a morphism of presheaves. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the ***presheaf kernel***

$$U \mapsto \ker\big(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big)$$

is a sheaf, called the ***kernel*** of φ and written $\ker\varphi$. The ***presheaf image***

$$U \mapsto \operatorname{im}\big(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)\big),$$

however, is not always a sheaf. The ***image*** of φ, denoted $\operatorname{im}\varphi$, is defined to be the sheaf associated to the presheaf image of φ.

A sheaf $\mathcal{F}$ over a space X is a ***subsheaf*** of a sheaf $\mathcal{G}$ if for every open set U in X the group $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$, and the inclusion map $i\colon \mathcal{F} \to \mathcal{G}$ is a presheaf morphism. If $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, the ***quotient sheaf*** is defined to be the sheaf associated to the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$.

A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is said to be ***injective*** if $\ker\varphi = 0$, and ***surjective*** if $\operatorname{im}\varphi = \mathcal{G}$.

Proposition 5.1.

(i) *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective if and only if the stalk map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is injective for every p.*

(ii) *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if the stalk map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective for every p.*

PROOF. Exercise (see [4, Exercise 1.2 (a),(b), p. 66]). □

In this proposition, neither (i) nor (ii) are true for morphisms of presheaves. A counterexample to (i) is $\mathcal{F} =$ the presheaf of Example 3.1 and $\mathcal{G} = 0$; a counterexample to (ii) is $\mathcal{F} = 0$ and $\mathcal{G} =$ the presheaf of Example 3.1. It is the truth of the proposition for sheaves that makes sheaves so much more useful than general presheaves.

Example 5.2. If the stalk $\mathcal{F}_p$ of a sheaf $\mathcal{F}$ vanishes for every $p \in X$, then by Proposition 5.1, the sheaf map $\mathcal{F} \to 0$ is both injective and surjective, since the stalk maps $\mathcal{F}_p \to 0_p$ are injective and surjective for all $p \in X$. Hence, $\mathcal{F}$ is isomorphic to the zero sheaf and has no nonzero global sections.

6. Exact Sequences of Sheaves

A sequence of sheaves

$$\cdots \longrightarrow \mathcal{F}^1 \xrightarrow{d_1} \mathcal{F}^2 \xrightarrow{d_2} \mathcal{F}^3 \xrightarrow{d_3} \cdots$$

on a topological space X is said to be ***exact*** at $\mathcal{F}^k$ if $\operatorname{im} d_{k-1} = \ker d_k$; the sequence is said to be ***exact*** if it is exact at every $\mathcal{F}^k$. By Proposition 5.1, the exactness of a sequence of sheaves on X is equivalent to the exactness of the sequence of stalk maps at every point $p \in X$. An exact sequence of sheaves of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \tag{6.1}$$

is said to be a ***short exact sequence***. The exactness of a sequence of groups is defined in the same way.

It is not too difficult to show that the exactness of the sheaf sequence (6.1) over a topological space X implies the exactness of the sequence of sections

$$0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \tag{6.2}$$

for every open set $U \subset X$, but the last map $\mathcal{F}(U) \to \mathcal{G}(U)$ need not be surjective. In fact, as we see in [1, Theorem 2.8], the cohomology $H^1(U, \mathcal{E})$ is a measure of the nonsurjectivity of the map of global sections $\mathcal{F}(U) \to \mathcal{G}(U)$.

Fix an open subset U of a topological space X. To every sheaf $\mathcal{F}$ on X, we can associate the abelian group $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$ of sections over U and to every sheaf map $\varphi \colon \mathcal{F} \to \mathcal{G}$, the group homomorphism $\varphi_U \colon \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$. This makes $\Gamma(U, \ )$ a functor from sheaves to abelian groups.

Example. Let $\mathcal{O}$ be the sheaf of holomorphic functions on the complex plane $\mathbb{C}$ and $\mathcal{O}^*$ the sheaf of nowhere-vanishing holomorphic functions on $\mathbb{C}$. For any open set $U \subset \mathbb{C}$, if $f \in \mathcal{O}(U)$, then $\exp 2\pi \mathrm{i} f \in \mathcal{O}^*(U)$. The kernel of the sheaf map $\exp 2\pi\mathrm{i}(\ )$ on any open set U consists of the holomorphic (hence locally constant) integer-valued functions on U. Hence, there is an exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0.$$

The surjectivity of $\mathcal{O} \to \mathcal{O}^*$ follows from the fact that if U is simply connected and $f \in \mathcal{O}^*(U)$, then $f(U)$ is a simply connected set in $\mathbb{C}$ not containing the origin, and hence $\log f$ is defined on U.

If U is the punctured plane $\mathbb{C} - \{0\}$, then the exponential map

$$\exp 2\pi\mathrm{i}(\ ) \colon \mathcal{O}(U) \to \mathcal{O}^*(U)$$

is not surjective, since it is not possible to define its inverse, $(1/2\pi\mathrm{i})\log$, on $\mathbb{C} - \{0\}$: for example, $\log z = \log(re^{\mathrm{i}\theta})$ must be defined as $(\log r) + \mathrm{i}\theta$, but the angle θ cannot be defined as a continuous function around a puncture at the origin.

A functor F from the category of sheaves on X to the category of abelian groups is said to be ***exact*** if it maps a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

to a short exact sequence of abelian groups

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G}) \to 0.$$

If instead one has only the exactness of

$$0 \to F(\mathcal{E}) \to F(\mathcal{F}) \to F(\mathcal{G}),$$

then F is said to be a ***left-exact functor***. Thus, the sections functor $\Gamma(U, \ )$ is left-exact but not exact.

7. Resolutions

Recall that $\underline{\mathbb{R}}$ is the sheaf of locally constant functions with values in $\mathbb{R}$ and $\mathcal{A}^k$ is the sheaf of C^∞ k-forms on a manifold M. For every open set U in M, the exterior derivative $d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$ induces a morphism of sheaves $d\colon \mathcal{A}^k \to \mathcal{A}^{k+1}$.

Proposition 7.1. *On any manifold M of dimension n, the sequence of sheaves*

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n \to 0 \tag{7.1}$$

is exact.

PROOF. Exactness at $\mathcal{A}^0$ is equivalent to the exactness of the sequence of stalk maps $\underline{\mathbb{R}}_p \to \mathcal{A}^0_p \xrightarrow{d} \mathcal{A}^1_p$ for all $p \in M$. Fix a point $p \in M$. Suppose $[f] \in \mathcal{A}^0_p$ is the germ of a C^∞ function $f\colon U \to \mathbb{R}$, where U is a neighborhood of p, such that $d[f] = [0]$ in $\mathcal{A}^1_p$. Then there is a neighborhood $V \subset U$ of p on which $df \equiv 0$. Hence, f is locally constant on V and $[f] \in \underline{\mathbb{R}}_p$. Conversely, if $[f] \in \underline{\mathbb{R}}_p$, then $d[f] = 0$. This proves the exactness of the sequence (7.1) at $\mathcal{A}^0$.

Next, suppose $[\omega] \in \mathcal{A}^k_p$ is the germ of a C^∞ k-form on some neighborhood of p such that $d[\omega] = 0 \in \mathcal{A}^{k+1}_p$. This means there is a neighborhood V of p on which $d\omega \equiv 0$. By making V smaller, we may assume that V is contractible. By the Poincaré lemma, ω is exact on V, say $\omega = d\tau$ for some $\tau \in \mathcal{A}^{k-1}(V)$. Hence, $[\omega] = d[\tau]$. □

In general, an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

on a topological space X is called a ***resolution*** of the sheaf $\mathcal{A}$. On a complex manifold M of complex dimension n, the analogue of the Poincaré lemma is the $\bar\partial$-Poincaré lemma [3, p. 25], from which it follows that for each fixed integer $p \geq 0$, the sheaves $\mathcal{A}^{p,q}$ of C^∞ (p,q)-forms on M give rise to a resolution of the sheaf Ω^p of holomorphic p-forms on M:

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n} \to 0. \tag{7.2}$$

The cohomology of the ***Dolbeault complex***

$$0 \to \mathcal{A}^{p,0}(M) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n}(M) \to 0$$

is by definition the ***Dolbeault cohomology*** of the complex manifold M. (For (p,q)-forms on a complex manifold, see [3].)

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