Analytic continuations of algebraic functions by means of Mellin-Barnes integrals

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Monodromy and Hypergeometric Functions
February 17-21, 2020
Galatasaray University, Istanbul

Outline

- The interaction of power series and Mellin-Barnes integrals.

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- The monodromy of the general algebraic function.

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- The interaction of power series and Mellin-Barnes integrals.
- The monodromy of the general algebraic function.
- The Mellin transform technique in study of solutions to polynomial systems.

General Algebraic Equation

Reduced equation

$$y^n + x_s y^{n_s} + \cdots + x_1 y^{n_1} - 1 = 0, \qquad (1)$$

where $0 = n_0 < n_1 < \cdots < n_s < n_{s+1} = n$, $x = (x_1, \ldots, x_s) \in \mathbb{C}^s$.

Principal solution

$$y(x) = y(x_1, \ldots, x_s),\ \ y(0) = 1;$$

denote it by $y_0(x)$.

Other branches

$$y_j(x) = \varepsilon^j y(x_1 \varepsilon^{jn_1}, \ldots, x_s \varepsilon^{jn_s}),\ \ j = 1, \ldots, n-1,\ \ \varepsilon = e^{\frac{2\pi i}{n}}.$$

Mellin's Formula (1921)

Hypergeometric type series for $y_0^\mu(x)$, $\mu > 0$:

$$y_0^\mu(x) = \frac{\mu}{n} \sum_{k \in \mathbb{Z}^s_\geqslant} \frac{(-1)^{|k|}\Gamma\left(\frac{\mu}{n} + \frac{n_1}{n}k_1 + \ldots + \frac{n_s}{n}k_s\right)}{k!\Gamma\left(\frac{\mu}{n} - \frac{n_1'}{n}k_1 - \cdots - \frac{n_s'}{n}k_s + 1\right)} x_1^{k_1}\ldots x_s^{k_s}, \quad (2)$$

where $k = (k_1, \ldots, k_s)$, $n_j' = n - n_j$, $k! = k_1! \cdot \ldots \cdot k_s!$.

Example

Consider the cubic equation

$$y^3 + xy - 1 = 0.$$

According to [Mellin, 1921] the principal solution can be represented by the integral

$$y_0(x) = \frac{1}{2\pi i} \int\limits_{\gamma + i\mathbb{R}} \frac{\frac{1}{3}\Gamma(\frac{1}{3} - \frac{1}{3}z)\Gamma(z)}{\Gamma(\frac{1}{3} + \frac{2}{3}z + 1)} x^{-z} dz, \tag{3}$$

which converges in the sector $S = \{x : |\arg x| < \frac{\pi}{3}\}$.

Example

Calculate the integral (3) as the sum of residues at poles $z = -k$, $k = 0, 1, \ldots$, of the function $\Gamma(z)$:

$$y_0(x) = \frac{1}{3}\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{1}{3} + \frac{1}{3}k)}{\Gamma(\frac{4}{3} - \frac{2}{3}k)k!} x^k, \quad |x| < \frac{3}{\sqrt[3]{4}}. \tag{4}$$

Calculate (3) as the sum of residues at poles $s = 1 + 3k$, $k = 0, 1, \ldots$, of the function $\Gamma(\frac{1}{3} - \frac{1}{3}z)$:

$$y(x) = \frac{1}{x}\sum_{k=0}^{\infty} \frac{\Gamma(1 + 3k)}{\Gamma(2 + 2k)k!} \frac{1}{(-x)^{3k}}, \quad |x| > \frac{3}{\sqrt[3]{4}}. \tag{5}$$

Example

The negative power of the principal solution can be represented by the integral

$$\frac{1}{y^{\mu}(x)} = \frac{1}{2\pi i} \int\limits_{\gamma + i\mathbb{R}} \frac{\frac{\mu}{3}\Gamma(z)\Gamma(\frac{\mu}{3} - \frac{z}{3})}{\Gamma(\frac{\mu}{3} + 1 + \frac{z}{3})}(-x)^{-z} dz, \ \ \mu > 0, \tag{6}$$

which converges in the sector $S' = \{x : \frac{\pi}{3} < \arg x < \frac{5\pi}{3}\}$.
Remark that

$$y(x) = -\frac{x}{y(x)} + \frac{1}{y(x)^2}. \tag{7}$$

$$y(x) = \frac{1}{2}(-x)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}k)}{\Gamma(\frac{3}{2} - \frac{3}{2}k)k!} \left(-\frac{1}{x}\right)^{\frac{3}{2}k}, \quad |x| > \frac{3}{\sqrt[3]{4}}. \tag{8}$$

Example

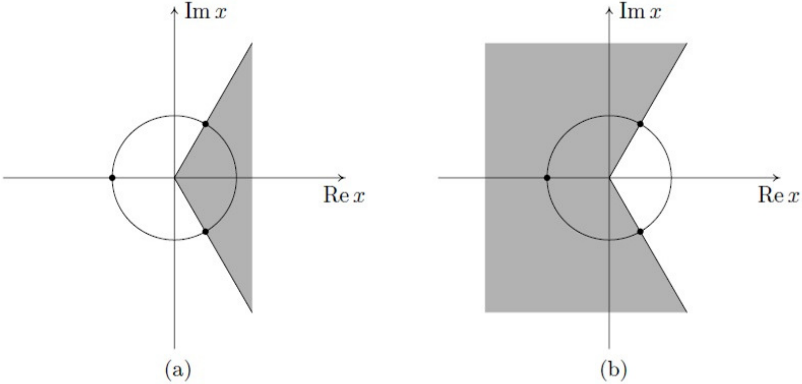

Fig. 1. Domains of convergence: (a) of the integral (3), and (b) of the integral (6).

Equation (1) is defined by the set of exponents $\{0,\, n_1, \dots, n_s, n\}$ of its monomials.

- A subdivision τ of the integer line segment $[0, n]$ is a collection of adjacent subsegments obtained by dividing the original segment at points of some subset of $\{n_1, \dots, n_s\}$.

Subdivision, Newton Polytope and Amoeba

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- The amoeba $\mathcal{A}$ of a polynomial is the image of its zero locus under the mapping

$$\mathrm{Log} : (x_1, \dots, x_s) \to (\log |x_1|, \dots, \log |x_s|).$$

Bijections

Let us introduce the following notations:

- $\nabla = \{\Delta(x) = 0\}$ is the discriminant locus of (1),
- $\mathcal{N}_\Delta$ is the Newton polytope of $\Delta(x)$,
- $\mathcal{A}_\nabla$ is the amoeba of the discriminant locus.

[Gelfand-Kapranov-Zelevinsky, 1994]

There exist bijections between the following sets:

$$\{\tau\} \leftrightarrow \{v_\tau\} \leftrightarrow \{E_\tau\},$$

where τ is a subdivision of the segment $[0, n]$, v_τ is a vertex of the Newton polytope $\mathcal{N}_\Delta$, E_τ is a component of the amoeba $\mathcal{A}_\nabla$ complement.

Puiseux Series

Theorem 1 [A.– Mikhalkin, 2012]

- For any ordered pair $p, q \in \{0, 1, \ldots, s+1\}$, the series $y_0^{\mu}(x)$ admits an analytic continuation in the form of the $(n_q - n_p)$-valued Puiseux series $y_{p,q}^{\mu}(x)$.

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- In each domain $\mathrm{Log}^{-1}(E_\tau)$, exactly n series $y_{p,q}^{\mu}(x)$ converge if one takes into account that each series is $(n_q - n_p)$-valued.

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- In each domain $\mathrm{Log}^{-1}(E_\tau)$, exactly n series $y_{p,q}^\mu(x)$ converge if one takes into account that each series is $(n_q - n_p)$-valued.

Remark: for the pair $p = 0$, $q = s+1$, the series coincides with the principal solution (2).

Discriminant Amoeba

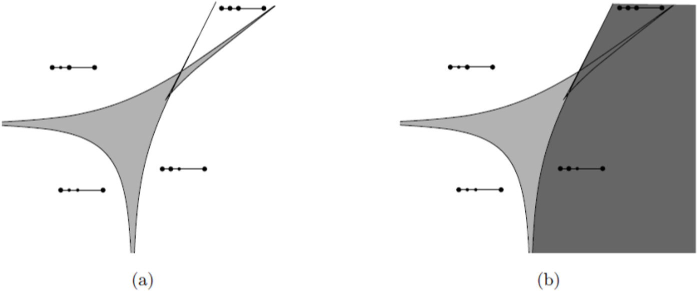

Fig. 2. (a) Amoeba of the discriminant of the equation $y^5 + x_2y^2 + x_1y - 1 = 0$ and the correspondence $\{\tau\} \leftrightarrow \{E_\tau\}$; (b) the image $\mathrm{Log}(D_{0,1})$ (the darker area).

Proof of Theorem 1

We consider the general algebraic equation

$$a_{s+1}y^{n} + a_s y^{n_s} + \cdots + a_1 y^{n_1} + a_0 = 0, \qquad (9)$$

with complex coefficients $a = (a_0, \ldots, a_{s+1})$. Its root $y(a)$ has a double homogeneity property.

It is enough to be able to solve equations of the type

$$r_{s+1}y^{n_{s+1}} + \cdots + y^{n_q} + \cdots - y^{n_p} + \cdots + r_0 y^{n_0} = 0 \qquad (10)$$

where $n_{s+1} = n$, $n_0 = 0$.

Let denote by $\tau_{p,q}(r)$ Taylor series solutions to (10) [Birkeland, 1927].

Series $\tau_{p,q}(r)$ may be turned into Puiseux series solutions to (9) by a monomial substitution $r = r(a)$.

Proof of Theorem 1

The equation (1) is a result of $(0, s+1)$ – dehomogenization of (9):

$$y^n + x_s y^{n_s} + \cdots + x_1 y^{n_1} - 1 = 0.$$

The series $\tau_{p,q}(r)$ (y(0)=1) may be turned into Puiseux series $y_{p,q}(x)$ by monomial substitutions $r = r(x)$.

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[Passare–Tsikh, 2004]

The domain of convergence $D'_{p,q}$ of the series $\tau_{p,q}(r)$ is a complete Reinhardt domain with the property that the domain $\mathrm{Log}(D'_{p,q})$ contains all the connected components of the set $\mathbb{R}^s \setminus \mathcal{A}_{\nabla_{p,q}}$ that are associated with subdivisions of $[0, n]$ containing the segment $[n_p, n_q]$.

The amoeba $\mathcal{A}_\nabla$ is a result of the affine transformation of the amoeba $\mathcal{A}_{\nabla_{p,q}}$.

Immediate Analytic Continuation

Consider the mapping

$$\mathrm{Arg} : (\mathbb{C} \setminus 0)^s_x \to \mathbb{R}^s_\theta$$

defined by the formula

$$\mathrm{Arg} : (x_1, \dots, x_s) \to (\arg x_1, \dots, \arg x_s),$$

where $\theta_j = \arg x_j$.
Introduce sectorial domains

$$S = \mathrm{Arg}^{-1} \left\{ |\theta_\nu| < \frac{\pi n_\nu}{n}, \nu \in I, |n_j \theta_k - n_k \theta_j| < \pi n_j, j, k \in I, j < k \right\},$$

$$S^{'} = \mathrm{Arg}^{-1} \left\{ |\theta_\nu + \pi| < \frac{\pi n^{'}_\nu}{n}, \nu \in I, |n^{'}_k (\theta_j + \pi) - n^{'}_j (\theta_k + \pi)| < \pi n^{'}_k, \right.$$

$$\left. j, k \in I, j < k \right\}, \quad I = \{1, \dots, s\}.$$

Immediate Analytic Continuation

Theorem 2 [A.– Mikhalkin, 2012]

The series $y_{0,q}$ is a result of analytic continuation of the principal solution y_0 from the domain $D_{0,s+1}$ to $D_{0,q}$ through the sectorial domain S, and $y_{p,s+1}$ is a result of analytic continuation of y_0 from $D_{0,s+1}$ to $D_{p,s+1}$ through the sectorial domain $S^{'}$.

Mellin's integral formula [A., 2007]

$$y_0^{\mu}(x) = \frac{1}{(2\pi i)^s} \int\limits_{\gamma+i\mathbb{R}^s} \frac{\frac{\mu}{n}\Gamma\left(\frac{\mu}{n} - \frac{1}{n}\langle\alpha, z\rangle\right)\Gamma(z_1)\ldots\Gamma(z_s)}{\Gamma\left(\frac{\mu}{n} + \frac{1}{n}\langle\beta, z\rangle + 1\right)} x^{-z} dz, \tag{11}$$

where $\alpha = (n_1, \ldots, n_s)$, $\beta = (n - n_1, \ldots, n - n_s)$ and

$$\gamma \in \{u \in \mathbb{R}^s_+ : \langle\alpha, u\rangle < \mu\}.$$

Integral (11) converges in the sectorial domain S.

Immediate Analytic Continuation

Integral representation for $1/y_0^{\mu}(x)$

$$\frac{1}{y_0^{\mu}(x)} = \frac{1}{(2\pi i)^s} \int\limits_{\gamma + i\mathbb{R}^s} \frac{\frac{\mu}{n}\Gamma\left(\frac{\mu}{n} - \frac{1}{n}\langle\beta, z\rangle\right)\Gamma(z_1)\ldots\Gamma(z_s)}{\Gamma\left(\frac{\mu}{n} + \frac{1}{n}\langle\alpha, z\rangle + 1\right)}(-x)^{-z}dz, \tag{12}$$

where

$$\gamma \in \{u \in \mathbb{R}_+^s : \langle\beta, u\rangle < \mu\}.$$

Integral (12) converges in the sectorial domain S'.

Remark: $y(x)$ satisfies the equation

$$y(x) = -\frac{x_s}{y^{n-n_s-1}(x)} - \cdots - \frac{x_1}{y^{n-n_1-1}} + \frac{1}{y^{n-1}(x)}.$$

Separating Cycle Principle [Tsikh, 1992]

Grothendieck-type integral

$$\frac{1}{(2\pi i)^s}\int\limits_{\Delta_g}\frac{h(z)dz}{f_1(z)\dots f_s(z)}, \qquad (13)$$

where

- $f=(f_1,\dots,f_s):\mathbb{C}^s\to\mathbb{C}^s$ is a holomorphic proper mapping,
- Δ_g is the skeleton of a polyhedron Π_g associated with a holomorphic proper mapping $g=(g_1,\dots,g_s):\mathbb{C}^s\to\mathbb{C}^s$,
- the polyhedron Π_g is $g^{-1}(G_1\times\dots\times G_s)$, $G_j\subset\mathbb{C}$,
- $\sigma_j=\{z:g_j(z)\in\partial G_j, g_k(z)\in G_k, k\neq j\}$, $j\in I$ are facets of Π_g,
- $D_j=\{f_j(z)=0\}$, $j\in I$ are polar divisors.

Separationg Cycle Principle

Definition

A polyhedron Π_g is said to be compartible with the set of divisors $\{D_j\}$, if for each j the corresponding facet σ_j of the polyhedron Π_g does not intersect the divisor D_j.

Assume that the intersection $Z = D_1 \cap \cdots \cap D_s$ is discrete.

Theorem

If the polyhedron Π_g is bounded and compartible with the family of polar divisors $\{D_j\}$, then the integral (13) equals to the sum of Grothendieck residues in the domain Π_g.

Remark: in the case of unbounded polyhedron, one needs to require that the integrand form should decrease sufficiently fast in Π_g. These conditions are described in the multidimensional abstract Jourdan lemma [Passare-Tsikh-Zhdanov, 1994].

Implementation of SCP for Mellin-Barnes Integrals

- We interpret the integration subspace $\gamma + i\mathbb{R}^s$ as the skeleton of a polyhedron.
- For $s > 1$ this subspace may serve as the skeleton for infinitely many polyhedra.

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- For $s > 1$ this subspace may serve as the skeleton for infinitely many polyhedra.
- We consider polyhedra of the type

$$\Pi_g = \{z \in \mathbb{C}^s : \operatorname{Re} g_j(z) < r_j, \, j \in I\},$$

  where $g_j(z)$ are linear functions with real coefficients.
- The polyhedron $\Pi_g = \pi + i\mathbb{R}^s$ where π is the s-dimensional simplicial cone in $\mathbb{R}^s$.

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- The polyhedron $\Pi_g = \pi + i\mathbb{R}^s$ where π is the s-dimensional simplicial cone in $\mathbb{R}^s$.
- The polar set for integrand in (11) consists of $s+1$ families of hyperplanes

$$D_j = \{z_j = -\nu, \, \nu \in \mathbb{Z}_{\geqslant}\}, \; D_{s+1} = \left\{\frac{\mu}{n} - \frac{1}{n}\langle \beta, z\rangle = -\nu, \nu \in \mathbb{Z}_{\geqslant}\right\}$$

Implementation of SCP for Mellin-Barnes Integrals

There exist $s+1$ cones $\pi_0, \pi_1, \ldots, \pi_s$ with vertices at γ such that

- $\Pi_0 = \pi_0 + i\mathbb{R}^s$ is compartible with the set of divisors $D_1, \ldots, D_s$,
- $\Pi_j = \pi_j + i\mathbb{R}^s$ is compartible with the set of divisors $D_1, \ldots [j] \ldots, D_s, D_{s+1}$.

According to the SCP we have $s+1$ residue formulas for the Mellin–Barnes integral (12) which give series $y_{p,s+1}$.

Horn-Kapranov parameterization of ∇ [Passare-Tsikh, 2004]

Consider the general algebraic equation

$$y^n + x_{n-1}y^{n-1} + \cdots + x_1 y - 1 = 0, \tag{14}$$

$y_0(x),\ y_0(0) = 1,\ y_j(x)$.
The discriminant locus ∇ of the equation (14) admits the parameterization $\Psi : \mathbb{CP}_s^{n-2} \to \mathbb{C}_x^{n-1}$ given by the formula

$$x_k = \Psi_k(s) = \frac{ns_k}{\langle \alpha, s\rangle}\left(-\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle}\right)^{\frac{k}{n}}, k = 1, \ldots, n-1, s \in \mathbb{CP}^{n-2}. \tag{15}$$

Let D be the convergence domain of the series $y_0(x)$. The surface

$$|x_k| = |\Psi_k(s)|,\ \ k = 1, \ldots, n, s \in \mathbb{R}_+^{n-1}, \tag{16}$$

gives the boundary $\partial|D|$ of the image $|D|$ on the Reinhardt diagramm.

Strings

Definition

The string $\mathcal{S}^{(j)}$ is said to be the surface

$$\mathcal{S}^{(j)} := \left\{ \Psi^{(j)}(s) : s \in \mathbb{R}_{+}^{n-1} \right\} \subset \nabla, \; j = 0, \ldots, n-1,$$

where $\Psi^{(j)}(s)$ is a branch of the parameterization (15) with the condition

$$\arg \left(-\frac{\langle \alpha, s \rangle}{\langle \beta, s \rangle} \right)^{\frac{1}{n}} = -\frac{\pi}{n}(1+2j).$$

Monodromy of the general algebraic function

Theorem 3 [Mikhalkin, 2015]

When extending through the boundary ∂D of the domain D, any branch $y_j(x)$ of the solution to (14) has the second order ramification on strings $\mathcal{S}^{(j)}$ and $\mathcal{S}^{(j-1)}$. The branch $y_j(x)$ going around the string $\mathcal{S}^{(j)}$ turns to the branch $y_{j+1}(x)$ and going around the string $\mathcal{S}^{(j-1)}$ turns to the branch $y_{j-1}(x)$.

Universal polynomial system

Consider a system of n polynomials

$$\sum_{\alpha \in A^{(i)}} a_\alpha^{(i)} y^\alpha = 0,\ i \in I := \{1, \ldots, n\}, \tag{17}$$

with unknown $y = (y_1, \ldots, y_n) \in (\mathbb{C} \setminus 0)^n$, variable coefficients $a_\alpha^{(i)}$, where $A^{(i)} \subset \mathbb{Z}^n$ are fixed subsets and $y^\alpha = y_1^{\alpha_1} \cdot \ldots \cdot y_n^{\alpha_n}$ is a monomial.

The solution $y(a) = (y_1(a), \ldots, y_n(a))$ is a multivalued algebraic vector-function with a polyhomogeneity property.

Reduced polynomial system

Consider the system of n polynomials

$$y^{\omega^{(i)}} + \sum_{\lambda \in \Lambda^{(i)}} x_{\lambda}^{(i)} y^{\lambda} - 1 = 0, \ \ i \in I, \tag{18}$$

with variable coefficients $x_{\lambda}^{(i)}$, where $\Lambda^{(i)} := A^{(i)} \setminus \{\omega^{(i)}, 0\}$ and a matrix $\omega = (\omega^{(i)})$ is nondegenerate.

Denote by Λ the disjunctive union of sets $\Lambda^{(i)}$, $\sharp\Lambda = N$, $\Lambda = \left(\lambda^1, \ldots, \lambda^N\right)$, φ_j are rows of the matrix Λ.

Mellin–Barnes Integral Representation for $y^d(x)$

Monomial function for the principal solution to (18), $y_i(0) = 1$

$$y^d(x) := y_1^{d_1}(x) \cdot \ldots \cdot y_n^{d_n}(x), \; d = (d_1, \ldots, d_n) \in \mathbb{R}_+^n. \qquad (19)$$

Theorem 4 [A., 2003],[Stepanenko, 2003]

$$\frac{1}{(2\pi i)^N} \int\limits_{\gamma + i\mathbb{R}^N} \prod_{j=1}^{n} \frac{\prod\limits_{\lambda \in \Lambda^{(j)}} \Gamma\left(z_\lambda^{(j)}\right) \Gamma\left(\frac{d_j}{\omega_j} - \frac{1}{\omega_j}\langle \varphi_j, z\rangle\right)}{\Gamma\left(\frac{d_j}{\omega_j} - \frac{1}{\omega_j}\langle \varphi_j, z\rangle + z_j + 1\right)} Q(z) x^{-z} \, dz$$

$$\gamma \in U = \{u \in \mathbb{R}_+^N : \langle \varphi_j, u\rangle < d_j, \; j \in I\},$$

$$Q(z) = \frac{1}{\det \omega} \det \left|\left| \delta_i^j (d_j - \langle \varphi_j, z\rangle) + \langle \varphi_j^{(i)}, z^{(i)} \rangle \right|\right|_{i,j \in I},$$

δ_i^j is the Kronecker symbol and $\omega = \operatorname{diag}[\omega_1, \ldots, \omega_n]$.

Convergence domains of the M-B integral

Consider the family of matrices

$$\left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right), \; \lambda^{(j)} \in \Lambda^{(j)}. \tag{20}$$

A minor of the matrix is said to be the principal minor if the set of numbers of distinguished rows coincides with the set of numbers of distinguished columns.

Theorem 5 [Kulikov, 2017]

The M-B integral associated to the solution to the system of polynomial equations has a nonempty convergence domain iff all the principal minors of all matrices (20)are positive.

Consider the system of equations

$$\begin{cases} y_1^4 + x_1 y_1^2 y_2 - 1 = 0, \\ y_2^4 + x_2 y_1 y_2^2 - 1 = 0. \end{cases} \tag{21}$$

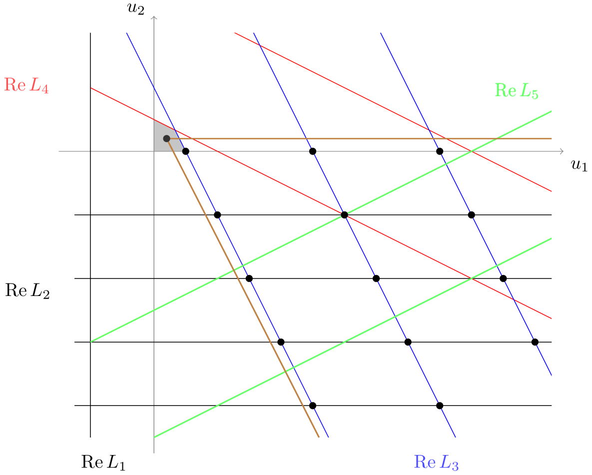
The Mellin-Barnes integral is as follows

$$\frac{1}{(2\pi i)^2} \int\limits_{\gamma + i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma\left(\frac{1}{4} - \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{1}{4} - \frac{1}{4}z_1 - \frac{1}{2}z_2\right)}{\Gamma\left(\frac{5}{4} + \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{5}{4} - \frac{1}{4}z_1 + \frac{1}{2}z_2\right)} \frac{(1 - z_1 - z_2)}{16} x^{-z} dz.$$

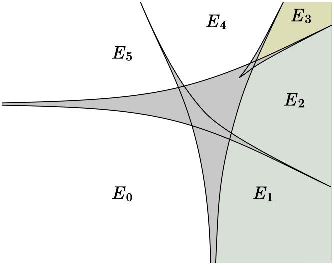
It converges in the sectorial domain $\mathrm{Arg}^{-1}(\Theta)$, where

$$\Theta = \left\{ |\theta_1| < \frac{\pi}{2},\ |\theta_2| < \frac{\pi}{2}, |2\theta_2 - \theta_1| < \frac{3\pi}{4}, |\theta_2 - 2\theta_1| < \frac{3\pi}{4} \right\}.$$

Polar divisors



Discriminant Amoeba



Theorem 6 [A-Kleshkova-Kulikov, 2020]

For any collection of n couples $\mu^{(i)}, \nu^{(i)} \in A^{(i)}$ with the nondegeneracy condition of the matrix

$$\varkappa = \left(\varkappa_j^{(i)}\right) = \left(\mu_j^{(i)} - \nu_j^{(i)}\right)$$

there exist an analytic continuation of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (18)

$$y^{\omega^{(i)}} + \sum_{\lambda \in \Lambda^{(i)}} x_\lambda^{(i)} y^\lambda - 1 = 0, \ i \in I,$$

in the form of the Puiseux series.