Hermitean invariants for the monodromy of A-hypergeometric systems

Frits Beukers (after Carlo Verschoor)

University of Utrecht

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The A-polytope

Hypergeometric functions à la Gel'fand-Kapranov-Zelevinsky (around 1988).
Start with a finite subset $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbb{Z}^r$ and $N > r$. We assume

- The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$
- There is a linear form h such that $h(\mathbf{a}_i) = 1$ for $i = 1, \ldots, N$.

The convex hull of A is called the *A-polytope*.
The $r \times N$-matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_N$ is called *A-matrix*.
Define a vector of parameters

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$$

Finite dimensional solution space

Consider the A-hypergeometric system $\mathscr{H}(A, \boldsymbol{\alpha})$.

Definition

Let $C(A)$ be the positive real cone generated by $\mathbf{a}_1, \ldots, \mathbf{a}_N$. We call the system *non-resonant* if $\boldsymbol{\alpha} + \mathbb{Z}^r$ has empty intersection with the faces of $C(A)$.

Theorem (GKZ), Adolphson

Let $Q(A)$ be the convex hull of A. Suppose the system is non-resonant. Then the solution space has $\mathbb{C}$-dimension $\mathrm{Vol}(Q(A))$ (standard simplex in $h(\mathbf{a}) = 1$ is normalized to 1).

Theorem (GKZ)

A non-resonant system has irreducible monodromy.

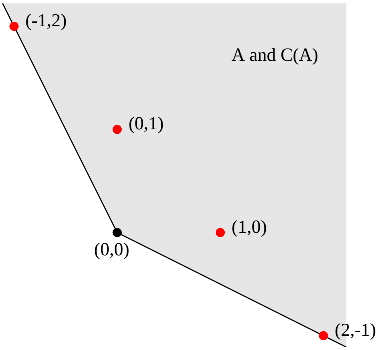
Horn G_3

Consider the Horn G_3-function

$$G_3(a,b,x,y) = \sum_{m,n\geq 0} \frac{(a)_{2n-m}(b)_{2m-n}}{m!n!} x^m y^n$$

We have

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{pmatrix}$$



The B-matrix of G_3

Recall

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{pmatrix}.$$

Lattice of relations is generated by the rows of

$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

We call this matrix the *B-matrix*.
We can recognize it from the expansion

$$\sum_{m,n\geq 0} \frac{(-x)^m(-y)^n}{\Gamma(m-2n-a+1)\Gamma(n-2m-b+1)\Gamma(m+1)\Gamma(n+1)}.$$

The columns of the B-matrix are denoted by $\mathbf{b}_1, \ldots, \mathbf{b}_N$.

Formal solutions

Let L be the lattice of integer relations between $\mathbf{a}_1, \ldots, \mathbf{a}_N$.
Choose $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ such that $\gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N = \boldsymbol{\alpha}$.
Then

$$\Phi = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1)\cdots\Gamma(l_N+\gamma_n+1)}$$

is a formal solution of the GKZ-system.

Note that $\boldsymbol{\gamma}$ is only determined modulo $L \otimes \mathbb{R}$, so we have some freedom for the choice of $\boldsymbol{\gamma}$.
More explicitly, we fix a $\boldsymbol{\gamma}_0$, then we can choose any $\boldsymbol{\gamma}$ of the form $\boldsymbol{\gamma}_0 + \boldsymbol{\mu} B$, where $\boldsymbol{\mu} \in \mathbb{R}^d$ is a row vector.
We use this freedom to derive different (convergent) series expansions for solutions of the A-hypergeometric system.

Power series solutions

Let $\mathscr{I}$ be the subsets J of $\{1, \ldots, N\}$ of size d such that $\Delta_J := |\det(\mathbf{b}_j)_{j \in J}| \neq 0$.
Consider again the formal solution

$$\Phi = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1) \cdots \Gamma(l_N+\gamma_n+1)}$$

Choose $J \in \mathscr{I}$ and choose $\boldsymbol{\gamma}$ such that $\gamma_j = 0$ for all $j \in J$.
Denote such a choice by $\boldsymbol{\gamma}^J$ and the corresponding solution by Φ_J.
Note that Φ_J is a summation over all $\mathbf{l}$ with $l_j \geq 0$ for all $j \in J$, a hyper-octant of summation. So we have a (twisted) powerseries.
In other words, Φ_J is $v_1^{\gamma_1^J} \cdots v_N^{\gamma_N^J}$ times a genuine power series (i.e. integers powers of the v_j).
We write $\boldsymbol{\gamma}^J = \boldsymbol{\gamma}_0 + \boldsymbol{\mu}^J B$, where $\boldsymbol{\mu}^J \in \mathbb{R}^d$ is a row vector.

Basis of local solutions

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y)=\sum_{m,n\geq 0}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n}\,x^m y^n.$$

Rewrite as

$$\sum_{m,n\geq 0}\frac{x^m y^n}{\Gamma(-\alpha-m-n+1)\Gamma(-\beta-m+1)\Gamma(-\beta'-n+1)\Gamma(\gamma+m)\Gamma(\gamma'+n)\Gamma(m+1)\Gamma(n+1)}.$$

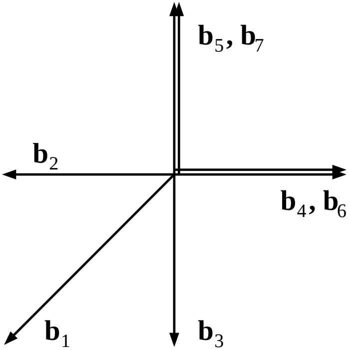
The B-matrix:

$$B=\begin{pmatrix}-1 & -1 & 0 & 1 & 0 & 1 & 0\\ -1 & 0 & -1 & 0 & 1 & 0 & 1\end{pmatrix}.$$

We sketch how to construct a basis of twisted power series solutions. The fan in $\mathbb{R}^d$ spanned by the column vectors $\mathbf{b}_i$ is called the *secondary fan*.

Basis of solutions

Secondary fan for Appell F_2:



Choose any vector $\boldsymbol{\rho}$ and find all pairs $\mathbf{b}_i, \mathbf{b}_j$ such that $\boldsymbol{\rho}$ is in the positive span of $\mathbf{b}_i, \mathbf{b}_j$. Denote the set of pairs by $\mathscr{I}(\boldsymbol{\rho})$. Then the solutions $\Phi_J, J \in \mathscr{I}(\boldsymbol{\rho})$ form a basis of local solutions.

Monodromy results

The one variable case:

- Levelt's theorem in Beukers-Heckman (1986)
- Methods using Mellin-Barnes integration (1908)

Several variable case:

- Appell F1, Lauricella F_D: E.Picard (1885), Deligne-Mostow (1986).
- Appell F2,F3: M.Kato (1995).
- Appell F4: K.Takano (1980), J.Kaneko (1981), M.Kato (1997).
- Aomoto system $E(3,6)$: K.Matsumoto, T.Sasaki, N.Takayama, M.Yoshida (1993).
- Lauricella F_A: K.Matsumoto, M.Yoshida (2010).
- Lauricella F_C: Y.Goto (2016).

Proposal

Approach via Mellin-Barnes integrals.

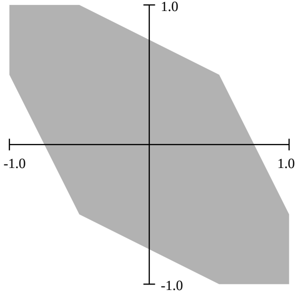
Assumption: The system $\mathscr{H}(\mathscr{A}, \boldsymbol{\alpha})$ is totally non-resonant, i.e. $\mathbb{Z}^r + \boldsymbol{\alpha}$ does not intersect the interior or exterior faces of the A-polytope.

The B-zonotope

Denote the columns of the B-matrix by $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \in \mathbb{R}^d$.
Define the *B-zonotope* by

$$Z_B = \left\{ \frac{1}{4} \sum_{j=1}^{N} \lambda_j \mathbf{b}_j \quad ; \quad -1 < \lambda_j < 1 \right\}$$

Picture for Horn G_3,



Convergence

Choose $\gamma_1, \ldots, \gamma_N$ such that $\gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N = \boldsymbol{\alpha}$.
Mellin-Barnes integral in higher dimension

$$M(\mathbf{v}) = \frac{1}{(2\pi i)^d} \int_{i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s}} d\mathbf{s}$$

Theorem

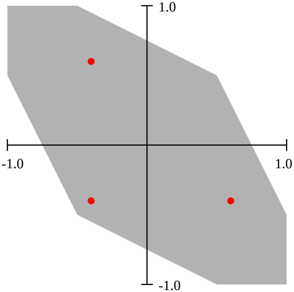
For $j = 1, \ldots, N$ let θ_j be an argument choice for v_j. Then the integral for $M(\mathbf{v})$ converges if

$$\theta_1 \mathbf{b}_1 + \theta_2 \mathbf{b}_2 + \cdots + \theta_N \mathbf{b}_N \in 2\pi Z_B.$$

Moreover, if $\gamma_j < 0$ for all j, then $M(\mathbf{v})$ is a solution of the hypergeometric system $A, \boldsymbol{\alpha}$.

B-zonotope for G_3

The Horn system G_3 has solution space of dimension 3. Consider the B-zonotope



Notice we have a basis of Mellin-Barnes solutions.

Transition matrices

Let $M_1, \ldots, M_D$ be a basis of Mellin-Barnes solutions and $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_D$ the corresponding points in the B-zonotope.

Choose a vector $\boldsymbol{\rho}$ in the secondary fan and let $\Phi_J, J \in \mathscr{I}(\boldsymbol{\rho})$ be the corresponding local basis of solutions. Let $\boldsymbol{\mu}^J \in \mathbb{R}^d$ be their corresponding vectors.

Theorem (FB)

The $D \times D$ transition matrix $X_{\boldsymbol{\rho}}$ between the local solutions and the Mellin-Barnes solutions is given by the entries

$$e^{2\pi i \boldsymbol{\mu}^J \cdot \boldsymbol{\tau}_j}, \quad J \in \mathscr{I}(\boldsymbol{\rho}), \quad j = 1, \ldots, D.$$

Restrictions

Hypotheses underlying the monodromy calculation.

1. We need a Mellin-Barnes basis of solutions, unfortunately this is not always possible.
2. Is the global monodromy group generated by the local contributions? Not clear.

A result in a different direction:

Theorem?

Let $M \subset GL_D(\mathbb{C})$ be the monodromy group of an irreducible A-hypergeometric system. Then there exists a non-trivial Hermitean matrix H such that $\overline{g}^t H g = H$ for all $g \in M$.

Euler integral

The existence of the invariant Hermitean form follows by general principles from the *Euler integral* representation.
Given $A, \boldsymbol{\alpha}$ define $f_A(\mathbf{v}, \mathbf{t}) = v_1\mathbf{t}^{\mathbf{a}_1} + \cdots + v_N\mathbf{t}^{\mathbf{a}_N}$, the A-polynomial.
Consider

$$I(A, \boldsymbol{\alpha}, v_1, \ldots, v_N) = \int_{\mathscr{C}} \frac{\mathbf{t}^{-\boldsymbol{\alpha}}}{1 - f_A(\mathbf{v}, \mathbf{t})} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_r}{t_r},$$

where $\mathscr{C}$ is a twisted r-cycle in the complement of $1 - f_A(\mathbf{v}, \mathbf{t}) = 0$.

Question: what is the signature of the Hermitean form?

Signature

Recall that to every $J \in \mathscr{I}$ we associate a vector $\boldsymbol{\gamma}^J \in \mathbb{R}^N$ such that $\gamma_1^J \mathbf{a}_1 + \cdots + \gamma_N^J \mathbf{a}_N = \boldsymbol{\alpha}$ and $\gamma_j^J = 0$ if $j \in J$.

Conjecture

Choose a local basis of solutions given by $\boldsymbol{\rho}$. Then the signature of the Hermitean form is determined by the signs of the products

$$\prod_{j \notin J} \sin(\pi \gamma_j^J), \qquad J \in \mathscr{I}(\rho).$$

Signature example

Signature in the case G_3. Take a triangulation of A,

$$\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\right\}$$

Write the parameter vector $(-a,-b)$ as linear combination of each of these pairs

$$a\begin{pmatrix}-1\\2\end{pmatrix}+(-b-2a)\begin{pmatrix}0\\1\end{pmatrix},\ -b\begin{pmatrix}0\\1\end{pmatrix}-a\begin{pmatrix}1\\0\end{pmatrix},\ (-a-2b)\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}2\\-1\end{pmatrix}$$

Then the signs of

- $\sin \pi a \cdot \sin \pi(-b-2a)$
- $\sin \pi(-a) \cdot \sin \pi(-b)$
- $\sin \pi(-a-2b) \cdot \sin \pi b$

determine the signature.

Hermitean form

We use the vector power notations $x^{\mathbf{k}} = (x^{k_1}, \ldots, x^{k_d})$ and $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \cdots z_d^{k_d}$.
Let $\boldsymbol{\tau} \in \mathbb{R}^d$. Let $\boldsymbol{\gamma}_0 \in \mathbb{R}^N$ be fixed such that $\gamma_{0,1}\mathbf{a}_1 + \cdots + \gamma_{0,N}\mathbf{a}_N = \boldsymbol{\alpha}$. Let $c_j = e^{2\pi i \gamma_{0,j}}$ for $j = 1, \ldots, N$.
Consider the differential form on $(\mathbb{C}^\times)^d$ given by

$$\omega(\boldsymbol{\tau}) = \frac{\mathbf{z}^{\boldsymbol{\tau}}}{(\mathbf{z}^{b_1} - c_1)\cdots(\mathbf{z}^{b_N} - c_N)} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}.$$

Let $J \in \mathscr{I}$. Then $e^{-2\pi i \boldsymbol{\mu}^J}$ is the solution of $\mathbf{z}^{\mathbf{b}_j} - c_j = 0$ for all $j \in J$.

Hermitean form

$$\omega(\boldsymbol{\tau}) = \frac{\mathbf{z}^{\boldsymbol{\tau}}}{(\mathbf{z}^{b_1} - c_1)\cdots(\mathbf{z}^{b_N} - c_N)} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_d}{z_d}.$$

Theorem (C.Verschoor, 2017)

Mellin-Barnes basis given by $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D$ in the B-zonotope.
Basis of local solutions given by $\boldsymbol{\rho}$.
Let G be the group generated by the local monodromies. Consider the $D \times D$-matrix with entries

$$\sum_{J \in \mathscr{I}(\boldsymbol{\rho})} \operatorname{res}_{e^{2\pi i \boldsymbol{\mu}^J}} \left(\omega(\boldsymbol{\tau}_i - \boldsymbol{\tau}_j)\right), \quad i, j = 1, \ldots, D.$$

Then this is the inverse of a G-invariant Hermitian matrix.