Generators of

Computing Fundamental Groups using Amoebas

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The A-hypergeometric system $\mathscr{H}(A;\beta)$ is a system of PDE's whose solutions are meromorphic functions on the affine space $\mathbb{C}^A$.

The principal symbol of $\mathscr{H}(A;\beta)$ is the *principal A-determinant* $E_A \subset \mathbb{C}^A$...

...and we have a monodromy representation

$$\mathcal{M}(\beta)\colon \pi_1(\mathbb{C}^A \setminus E_A; w) \to \mathrm{M}_A(\beta).$$

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Monodromy of *A*-hypergeometric functions

By *Frits Beukers* at Utrecht

Abstract. Using Mellin–Barnes integrals we give a method to compute elements of the monodromy group of an A-hypergeometric system of differential equations. The method works under the assumption that the A-hypergeometric system has a basis of solutions consisting of Mellin–Barnes integrals. Hopefully these elements generate the full monodromy group, but this has only been verified in some special cases.

$$\mathcal{M}(\beta)\colon \pi_1(\mathbb{C}^A \setminus E_A; w) \to \mathrm{M}_A(\beta).$$

Problems:

- Does there exist a Mellin–Barnes basis of solutions?
- Is the fundamental group generated by *amoebic paths*?

Series	M–B	$\alpha_1 = \pi_1$
F_1, G_2	T	T
F_2, F_3, H_2	T	T
G_1, H_3, H_6	T	T
H_1	T	T
H_4, H_7	T	T
G_3	T	T
H_5	T	T
F_4	F	F

$$\mathbb{R}^k \xleftarrow{\text{Log}} (\mathbb{C}^*)^k \xrightarrow{\text{Arg}} (S^1)^k$$

Definition

Let $Z \subset (\mathbb{C}^*)^k$. Then, the *amoeba* of Z is the projection

$$\mathcal{A}(Z) = \text{Log}(Z),$$

and the *coamoeba* of Z is the projection

$$\mathcal{C}(Z) = \text{Arg}(Z).$$

Definition 1.4. The *amoeba* of a Laurent polynomial f is the subset $\log(Z_f) \subset \mathbf{R}^k$.

This name is motivated by the following typical shape of $\log(Z_f)$ in two dimensions (see Figure 16).

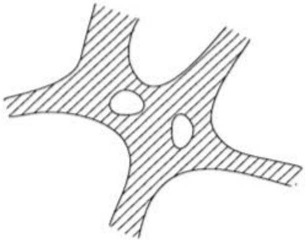


Figure 16. Amoeba

This shape is peculiar because of the thin "tentacles" going off to infinity. A bit later we shall give rigorous statements showing that the behavior of $\log(Z_f)$ is indeed typical. But first we relate the amoeba to the problem of finding Laurent series expansions for the rational function $1/f(x)$. Recall the general properties of Laurent series in several variables and their regions of convergence, see e.g., [Kr].

from [Gelfand–Kapranov–Zelevinsky]

J Geom Anal (2013) 23:24–46
DOI 10.1007/s12220-011-9235-7

Mellin Transforms of Multivariate Rational Functions

Lisa Nilsson · Mikael Passare

Theorem 4 *For any connected component E of the coamoeba complement $\mathbb{R}^n \setminus \overline{\mathcal{A}'_f}$ there is an integral representation*

$$\frac{1}{f(z)} = \int_{\sigma + i\mathbb{R}^n} M^E_{1/f}(s) z^{-s}\, ds, \tag{19}$$

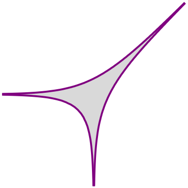
which converges for all z in the domain $\mathrm{Arg}^{-1}(E)$. Here σ is an arbitrary point in $\mathrm{int}\,\Delta_f$ *and*

$$M^E_{1/f}(s) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{f(z)} \frac{dz}{z} = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \frac{e^{\langle s, x+i\theta \rangle}}{f(e^{x+i\theta})}\, dx, \tag{20}$$

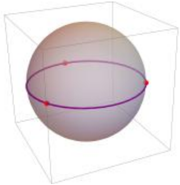
with θ being an arbitrary point in the component E.

Amoebas are one part algebraic and one part combinatorial.

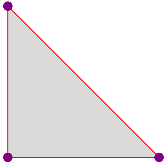
$$f(z_1, z_2) = 1 + z_1 + z_2$$



$\mathcal{A}$



Z



$\mathcal{N}$

Theorem (Forsberg–Passare–Tsikh)

Let $f \in \mathbb{C}[z_1, \ldots, z_k]$, *with Newton polytope* $\mathcal{N}$. *Then, there is a map*

$$\mathrm{ord}_f \colon \pi_0\big(\mathbb{R}^n \setminus \mathcal{A}\big) \hookrightarrow \mathcal{N} \cap \mathbb{Z}^k.$$

AMOEBAS, MONGE-AMPÈRE MEASURES, AND TRIANGULATIONS OF THE NEWTON POLYTOPE

MIKAEL PASSARE and HANS RULLGÅRD

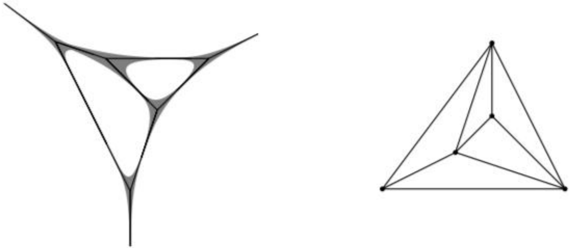
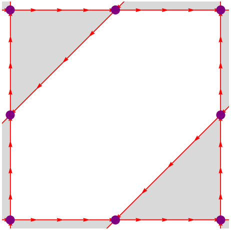


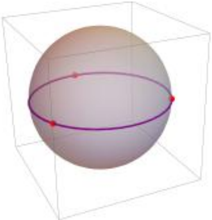
Figure 1. Amoeba of the polynomial $1 + z_1^5 + 80z_1^2 z_2 + 40z_1^3 z_2^2 + z_1^3 z_2^4$ (shaded) together with its spine (solid) and the dual triangulation of the Newton polytope

Coamoebas are one part algebraic and one part combinatorial.

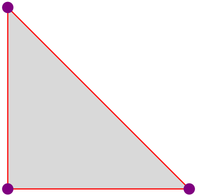
$$f(z_1, z_2) = 1 + z_1 + z_2$$



$\mathcal{C}$



Z



$\mathcal{N}$

The A in “A-hypergeometric” stands for the support set of the quasi-homogenization $f_h(z_0, z) = z_0 f(z)$.

Gale duality: $\quad AB = 0.$

$$f(z_1, z_2) = 1 + z_1^5 + 80\, z_1^2 z_2 + 40\, z_1^3 z_2^2 + z_1^3 z_2^4.$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 2 & 3 & 3 \\ 0 & 0 & 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ -4 & -10 \\ 0 & 5 \\ 1 & 0 \end{bmatrix}$$

The *dimension* is $n = \operatorname{rank}(A) - 1$ and the *codimension* is $m = \operatorname{rank}(B)$.

Let $A = \{\alpha_1, \ldots, \alpha_N\}$ with Gale dual $B^\top = \{\beta_1, \ldots, \beta_N\}$, and consider the Zonotope

$$\mathcal{Z} = \left\{ \frac{\pi}{2} \sum_{i=1}^{N} \lambda_i \, \beta_i \;\middle|\; |\lambda_i| \leq 1 \right\} \subset \mathbb{R}^m.$$

Theorem (F. & Johansson)

Let $f \in \mathbb{C}[z_1, \ldots, z_k]$, with dual zonotope $\mathcal{Z}$. Then, there is a map

$$\operatorname{cor}_f \colon \mathcal{Z}^\circ \cap \big(\xi_f + 2\pi\mathbb{Z}^m\big) \hookrightarrow \pi_0\big((S^1)^n \setminus \overline{\mathcal{C}}\big).$$

$$\mathbb{R}^k \xleftarrow{\text{Log}} (\mathbb{C}^*)^k \xrightarrow{\text{Arg}} (S^1)^k$$

Let $Z \subset (\mathbb{C}^*)^k$ be fixed. We write

$$\mathcal{A} = \mathcal{A}(Z) \quad \text{and} \quad \mathcal{C} = \mathcal{C}(Z).$$

For $x \in \mathbb{R}^k$, let

$$\mathcal{C}_x = \mathcal{C}(Z \cap \text{Log}^{-1}(x)).$$

For $t \in (S^1)^k$, let

$$\mathcal{A}_t = \mathcal{A}(Z \cap \text{Arg}^{-1}(t)).$$

Each $x \in \mathbb{R}^k$ defines an inclusion

$$\iota_x \colon (S^1)^k \setminus \mathcal{C}_x \to (\mathbb{C}^*)^k \setminus Z$$

by $t \mapsto (x, t)$. Hence, we get an inclusion

$$\pi_1\big((S^1)^k \setminus \mathcal{C}_x; t_1, t_2\big) \to \pi_1\big((\mathbb{C}^*)^k \setminus Z; z_1, z_2\big)$$

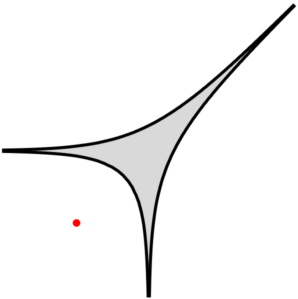
where $z_1 = (x, t_1)$ and $z_2 = (x, t_2)$. We call en element in the former fundamental group an *angular move*.

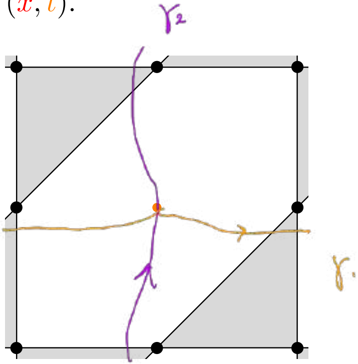
This situation is the most interesting if $x \in \mathbb{R}^k \setminus \mathcal{A}$, so that $\mathcal{C}_x = \varnothing$, and $t_1 = t_2 = t$. Then, we obtain an embedding

$$\mathbb{Z}^k \simeq \pi_1\big((S^1)^k; t\big) \to \pi_1\big((\mathbb{C}^*)^k \setminus Z; z\big),$$

where $z = (x, t)$.

Consider $\pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big)$, where $z = (x, t)$.





That is, there is a canonical embedding

$$\varphi\colon \pi_1\big((S^1)^2; t\big) \hookrightarrow \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big).$$

Each $t \in (S^1)^k$ defines an inclusion

$$\iota_t \colon \mathbb{R}^k \setminus \mathcal{A}_t \to (\mathbb{C}^*)^k \setminus Z$$

by $x \mapsto (x, t)$. Hence, we get an inclusion

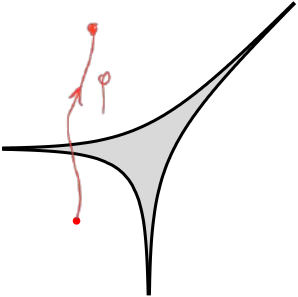
$$\pi_1\big(\mathbb{R}^k \setminus \mathcal{A}_t; x_1, x_2\big) \to \pi_1\big((\mathbb{C}^*)^k \setminus Z; z_1, z_2\big)$$

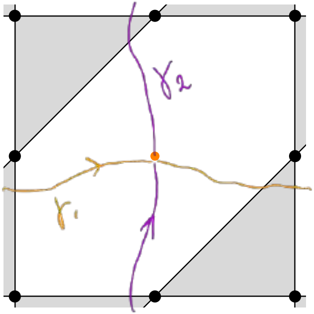
where $z_1 = (x_1, t)$ and $z_2 = (x_2, t)$. We call en element in the former fundamental group a *modular move.*

This situation is not very interesting if $t \in (S^1)^k \setminus \mathcal{C}$, so that $\mathcal{A}_t = \varnothing$, and $x_1 = x_2 = x$, for then

$$\pi_1\big(\mathbb{R}^k; x\big) = 0.$$

Consider $\pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big)$, where $z = (x, t)$.

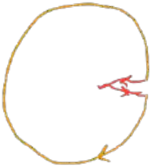




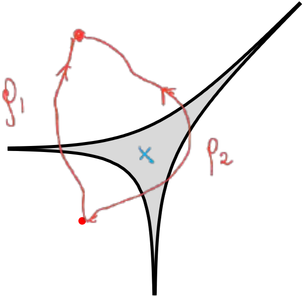
That is, each $E_j \in \pi_0\big(\mathbb{R}^2 \setminus \mathcal{A}\big)$ gives an embedding

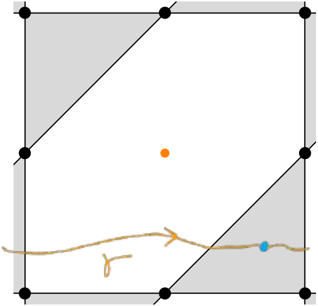
$$\varphi_j \colon \pi_1\big((S^1)^2; t\big) \hookrightarrow \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big).$$

$\gamma \mapsto \rho^{-1}\gamma\rho$



Consider $\pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big)$, where $z = (x, t)$.





That is, each $E_j \in \pi_0\big(\mathbb{R}^2 \setminus \mathcal{A}\big)$ and $\rho_{j\ell} \in \pi_0(\mathbb{R}^2 \setminus \mathcal{A}_t)$ gives an embedding

$$\varphi_{j\ell} \colon \pi_1\big((S^1)^2; t\big) \hookrightarrow \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big).$$

Definition

The *amoebic fundamental group* is the subgroup

$$\alpha_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big) = \big\langle \mathrm{Im}\,(\varphi_1), \ldots, \mathrm{Im}\,(\varphi_K)\big\rangle \subset \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big).$$

Question

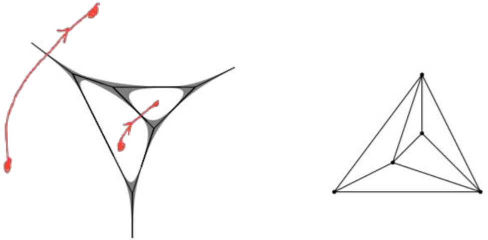
Under what conditions is

$$\alpha_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big) = \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big)?$$

Proposition

Let $\ell \subset \mathbb{R}^2$ be a piecewise smooth curve segment from $x_1 \in E_1$ to $x_2 \in E_2$ intersecting the spine $\mathcal{S}$ of the amoeba in an edge with primitive integer tangent vector $k = (k_1, k_2)$. Then,

$$\gamma_{11}^{k_1}\gamma_{12}^{k_2} = \gamma_{21}^{k_1}\gamma_{22}^{k_2}.$$



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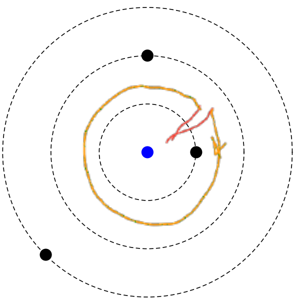
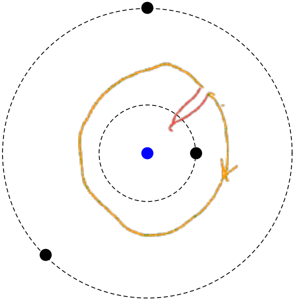




Proposition

If f is a univariate polynomial, then

$$\alpha_1\big(\mathbb{C}^* \setminus Z_f; z\big) = \pi_1\big(\mathbb{C}^* \setminus Z_f; z\big)$$

if and only if all roots of f are separated in moduli. □

Definition

An amoeba $\mathcal{A}$ is *maximal* if its order map is surjective.

Conjecture

If the amoeba $\mathcal{A}$ is maximal, then

$$\alpha_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big) = \pi_1\big((\mathbb{C}^*)^2 \setminus Z_f; z\big).$$

...but if $\mathcal{A}$ is maximal then typically Z_f is nodal and intersects the boundary of $(\mathbb{C}^*)^2$ transversally...

A characterization of *A*-discriminantal hypersurfaces in terms of the logarithmic Gauss map

M. M. Kapranov *

Theorem 1.3. *Let $G=(\mathbb{C}^*)^m$ be an algebraic torus, $Z \subset G$ – an algebraic irreducible hypersurface. The Gauss map $\gamma_Z : Z \to \mathbb{P}^{m-1}$ is birational if an only if there exist $n>0$, a finite subset $A \subset \mathbb{Z}^{n-1}$ as above and an isomorphism of tori $G \to T(L_A)$ taking Z to the reduced A-discriminantal hypersurface $\tilde{\nabla}_A$.*

Corollary

Let A be a configuration of codimension $m=1$, and let D_A denote the reduced principal A-determinant. Then,

$$\alpha_1\big(\mathbb{C}^* \setminus D_A; w\big) = \pi_1\big(\mathbb{C}^* \setminus D_A; w\big).$$

Proposition

Let D_A denote the A-discriminant and E_A the principal A-determinant. If $\mathcal{C} \neq (S^1)^2$ and

a) $\operatorname{Log}\colon E_A \to \mathcal{A}$ *is 2-1 outside of the preimage of the contour*

b) $\operatorname{Arg}\colon E_A \to \mathcal{C}$ *is 1-1 outside of the preimage of the shell*

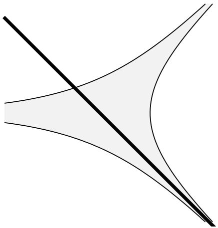
c) *The lattice width of $\mathcal{N}(D_A)$ is at most 1.*

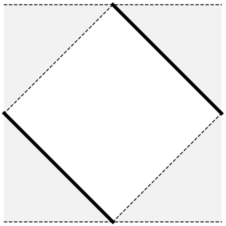
then

$$\alpha_1\big((\mathbb{C}^*)^2 \setminus E_A; z\big) = \pi_1\big((\mathbb{C}^*)^2 \setminus E_A; z\big)$$

Series	Principal A-determinant
F_1 and G_2	$(1-z_1)(1-z_2)(z_1-z_2)$
F_2, F_3, and H_2	$(1-z_1)(1-z_2)(1-z_1-z_2)$
G_1, H_3, and H_6	$(1-4z_1z_2)(1-z_1-z_2)$

Series	Principal A-determinant
H_4 and H_7	$(1 + 4z_1z_2)(1 - 2z_2 + z_2^2 + 4z_1z_2)$
H_1	$(1 - z_1)(1 - z_2)(1 - 2z_2 + z_2^2 + 4z_1z_2)$





Consider the case of $n = m = 1$. That is,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & k & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d-k \\ -d \\ k \end{bmatrix},$$

with $\gcd(k,d) = 1$. We consider *univariate trinomials* of degree d:

$$f(z) = w_0 + w_1\, z^k + w_2\, z^d.$$

Theorem

Let $A = \{0, k, d\}$ with $\gcd(k,d) = 1$. Then, the braid map

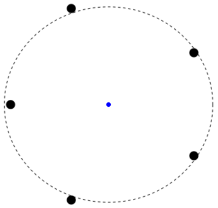
$$\pi_1\big((\mathbb{C}^*)^3 \setminus D_A; w\big) \to \mathcal{CB}_d,$$

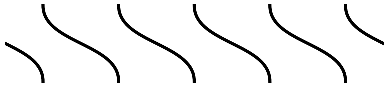
into the cyclic braid group on d strands, is surjective.

$$f(z) = w_0 + w_1\, z^k + w_2\, z^d$$

Choose a basepoint with
$|w_1| \ll \min\big(|w_0|, |w_2|\big)$

$$w(t) = \big(w_0, w_1, e^{2\pi i t} w_2\big)$$





The discriminant of the normalized polynomial

$$f(z) = 1 + w\, z^k + z^d$$

is a d-fold covering of the A-discriminant.

Theorem

Let $A = \{0, k, d\}$ *with* $\gcd(k, d) = 1$*. Then, the braid map*

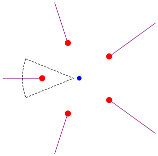
$$\pi_1\big((\mathbb{C}^*)^3 \setminus D_A; w\big) \to \mathcal{CB}_d,$$

into the cyclic braid group on d *strands, is surjective.*

Proof. The zonotope $\mathcal{Z}$ is an interval of length $2\pi d$. Recall,

$$\mathrm{cor}_f \colon \mathcal{Z}^\circ \cap \big(\xi_f + 2\pi\mathbb{Z}\big) \hookrightarrow \pi_0\big(S^1 \setminus \mathcal{C}\big),$$

where $\xi_f = d \arg_\pi(w)$. □

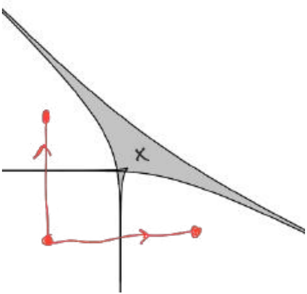


Corollary

If $A = \{0, k_1, \ldots, k_m, d\}$ *with* $\gcd(k_1, \ldots, k_m, d) = 1$, *then* $\pi_1\big((\mathbb{C}^*)^m \setminus D_A; w\big) \simeq \mathbb{Z} \times \mathcal{CB}_d$.

N.B., there might be no Mellin–Barnes basis of solutions.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 6 \end{bmatrix}$$



Corollary

If $A = \{0, k_1, \ldots, k_m, d\}$ *with* $\gcd(k_1, \ldots, k_m, d) = 1$, *then* $\pi_1\big((\mathbb{C}^*)^m \setminus D_A; w\big) \simeq \mathbb{Z} \times \mathcal{CB}_d$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{bmatrix}$$

The subset $\{0, k_i, d\}$ gives a braid which has $\gcd(k_i, d)$ equidistributed fundamental flips. Since $\gcd(k_1, \ldots, k_m, d) = 1$, these braids generates the full cyclic braid group $\mathcal{CB}_d$.

Given

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & k_1 & \dots & k_m & d \end{bmatrix}$$

with

$$\gcd(k_1, \dots, k_m, d) = 1,$$

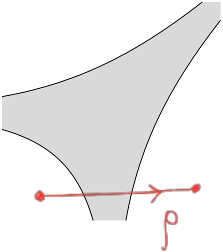
the monodromy group of the A-hypergeometric system, for non-resonant parameters, depends only on d.

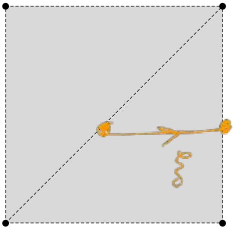
In particular, we can replace A by the codimension one configuration

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & d \end{bmatrix}$$

without changing the monodromy group. The latter admits a Mellin–Barnes basis of solutions.

Apell's F_4:

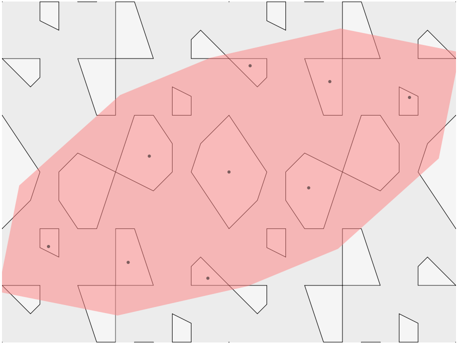




$$E(z_1, z_2) = 1 - 2z_1 + z_1^2 - 2z_2 - 2z_1z_2 + z_2^2$$

The missing cycle is $\xi^{-1}\rho^{-1}\xi\rho$

Thank you!



$$f(z_1, z_2) = 1 + z_1^3 + z_2^2 + z_1^3 z_2 + z_1^2 z_2^2$$