Descent Theory for covers

Anna Cadoret

G.A.G.T. summer school - Istanbul, June 8th - 19th, 2009

The aim of this lecture is to provide a survey of descent theory with emphasis to its application to the regular inverse Galois problem. This, essentially, will amount to study descent over the small etale site of the spectrum of a field.

I will first recall briefly the classical formalism of descent (fibered categories, pseudo-functors, descent data, prestacks, stacks *etc.*) and give some classical examples among which the stacks:

1. $\mathcal{M}_g$ of proper, smooth, geometrically connected genus $g \neq 1$ curves;
2. $\mathcal{H}_G(\mathbf{C})$ of G-covers of the projective lines with group G and inertia canonical invariant $\mathbf{C}$;
3. $\mathcal{H}_G(\mathbf{C})$ of genus g G-curves with group G and whose resulting G-cover has inertia canonical invariant $\mathbf{C}$,

I will describe explicitly descent along torsors and, in particular, what occurs over the small etale site of the spectrum of a field, where one recovers the so-called "Weil's cocycles conditions".

I will then introduce the notion of coarse moduli space of, say, an etale stack $\mathcal{X}$ over a field k. Roughly, it can be regarded as a sheaf X together with a stack morphism $\mathcal{X} \to X$. Solving the regular inverse Galois problem for a finite group G over k amounts to showing that $\mathcal{H}_G(\mathbf{C})(k) \neq \emptyset$ (for some inertia canonical invariant $\mathbf{C}$) and, the method to tackle this problem relies on two steps:

1. (Arithmetico-geometrical step) Show that $X(k) \neq \emptyset$;
2. (Descent step) Given $x \in X(k)$, find conditions so that x lies in the image of $\mathcal{X}(k) \to X(k)$.

When $\mathcal{X} = \mathcal{M}_g$, $\mathcal{H}_G(\mathbf{C})$, $\mathcal{H}_G(\mathbf{C})$, the coarse moduli spaces X are schemes and one can catch some arithmetico-geometric informations about them which, in some cases, are enough to ensure that $X(k) \neq \emptyset$. After reviewing some of the classical facts about this, I will focus on the descent step.

I will mostly describe two approaches:

1. Cohomological approach: For each $x \in X(k)$, a "cohomological obstruction" ω_x which lives in some cohomology pointed set or group for Γ_k with the property that ω_x becomes trivial over some field extension K/k if and only if x lies in the image of $\mathcal{X}(K) \to X(K)$.
2. Arithmetico-geometrical approach: For each $x \in X(k)$, a "descent variety" $V_x \to k$ for x with the property that the fiber of $V_x \to k$ over a field extension K/k are the elements of $\mathcal{X}(K)$ lying above x. This construction has a global variant over X, that is one can construct a X-scheme $V \to X$ such that, for any closed point $x \in X$, $V_x \to k(x)$ is a descent variety for x.

I will try and describe carefully these constructions and give several applications to regular inverse Galois theory such as:

1. Regular inverse Galois problem over number fields (for some special groups G:));
2. Regular inverse Galois problem over certain classes of fields (cohomological dimension ≤ 1, totally p-adic or totally real fields *etc*);
3. Local-global principles.