Middle convolution and the Inverse Galois Problem.

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Abstract: The middle convolution $\mathrm{MC}_\chi(V)$ of a local system V on the punctured affine line is obtained by considering a subsheaf of the higher direct image with compact supports of a twist of V (depending on a character $\chi : \pi_1(\mathbf{G}_m) \to \bar{\mathbf{Q}}_\ell^\times$) along the addition map $\pi : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$. Nicholas Katz has shown that all rigid local systems (e.g., hypergeometric sheaves) arise by iteratively middle convoluting and tensoring rank one local systems. This yields linear groups $\mathrm{GL_n}(\mathbf{F}_q)$ as Galois groups over $\mathbf{Q}(t)$ (for $q < n$ odd). Since MC_χ commutes with the braiding action, one can apply the middle convolution also to nonrigid local systems in a meaningful way. This yields many orthogonal and symplectic groups as Galois groups over $\mathbf{Q}(t)$. Using a motivic interpretation of MC_χ in the étale setting it is even possible to realize special linear groups over $\mathbf{Q}(t)$.

Course 1: The middle convolution as variation of parabolic cohomology.

We recall the general theory of variations of parabolic cohomology in the setup of [DW]. The relation of singular cohomology with group cohomology makes this concept accessible to explicit calculation. Since the middle convolution $\mathrm{MC}_\chi(V)$ of a local V system is a special case of a variation of parabolic cohomology, the monodromy matrices of MC_χ can then be calculated explicity in terms of the monodromy matrices of V. We explain the $\mathrm{GL_n}(\mathbf{F}_q)$ realizations mentioned in the abstract.

Course 2: Hurwitz spaces and the middle convolution The explicit monodromy matrices for $\mathrm{MC}_\chi(V)$ can be used to show that MC_χ commutes with the braiding action on (monodromy-)tuples. We recall the theory of Hurwitz spaces (cf. [V], [MM]) and show, how one can use MC_χ to jump from one Hurwitz space to another. In this way we can transfer rational points from a Hurwitz space of a small group (e.g., a dihedral group) to a Hurwitz space of a symplectic or orthogonal group. Since rational points on Hurwitz spaces correspond to Galois realizations this yields many classical groups as Galois groups over $\mathbf{Q}(t)$ ([D],[DR]).

Course 3: Motives and the middle convolution in the étale setting. We give the definition of MC_χ for lisse étale sheaves. One can prove the existence of an underlying family of motives M_s such that every specialization of $\mathrm{MC}_\chi(V)$ at rational points $s \in \mathbf{A}^1(\mathbf{Q})$ yields a compatible system of Galois representations $(\rho_\ell^s : G_{\mathbf{Q}} \to \mathrm{GL}_n(\bar{\mathbf{Q}}_\ell))$ which is formed by the the étale realizations of M_s. This can be used to prove the existence of interesting motives and to realize special linear groups over $\mathbf{Q}(t)$.

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