Ample Fields

This is an outline of a mini course organized by Moshe Jarden in the GTEM Summer School, "Geometry and Arithmetic around Galois Theory" to be held in Galatasaray University, Istanbul from the 8th to the 19th of June, 2009. The mini course will split into four sessions, given by four lecturers.

1. Moshe Jarden: Definitions, Examples, and Properties of Ample Fields.

A field K is said to be **ample** if every absolutely irreducible curve C defined over K with a simple K-rational point has infinitely many K-rational points. Examples for ample fields are PAC fields, Henselian fields, and fields whose absolute Galois group is a pro-p group for some prime number p. If E is a function field of one variable over an ample field K, then every finite split embedding problem over K is solvable over E. Consequently, every countable Hilbertian PAC field has a free absolute Galois group of countable rank.

Literature: [Jar10, Chapter 5].

2. Dan Haran: Algebraic Patching.

Let E be a field, G a finite group, and I a finite set. For each $i \in I$ we are given a subgroup G_i of G, a Galois extension F_i of E with Galois group G_i, and a field extension P_i of E, all contained in a common field P. Suppose $G = \langle G_i \mid i \in I\rangle$, $\bigcap_{i\in I} P_i = E$, $P_i' = \bigcap_{j\neq i} P_i$ contains F_i for each $i \in I$, and for every positive integer n, every matrix $A \in \mathrm{GL}_n(P)$ can be decomposed as $A = A_i A_i'$ with $A_i \in \mathrm{GL}_n(P_i)$ and $A_i' \in \mathrm{GL}_n(P_i')$. Then E has a Galois extension F with Galois group G.

Moreover, suppose E is a Galois extension of a field E_0 with Galois group Γ. Suppose Γ acts on E, G, I, and P in a compatible way. Then F is a Galois extension of E_0, $\mathrm{Gal}(F/E_0) = \Gamma \ltimes G$, and the restriction res: $\mathrm{Gal}(F/E_0) \to \mathrm{Gal}(E/E_0)$ coincides with the map $\pi\colon \Gamma \ltimes G \to \Gamma$ on the first factor.

Literature: [Jar10, Chapter 1].

3. Wulf-Dieter Geyer: Fields of Convergent Power Series.

Let $\hat{K}$ be a field complete under an ultra-metric absolute value $|\ |$ (e.g. $\hat{K} = K((t))$ and K is an arbitrary field), x a variable, and $E = \hat{K}(x)$. Let G be a finite group and let G_i, with i ranging on a finite set I, be subgroups of G whose orders are powers of prime numbers. For each $i \in I$ let $c_i \in I$ such that $c_i \neq c_j$ if $i \neq j$ and let $r \in E^\times$ such that $|r| \leq |c_i - c_j|$ for $i \neq j$. Set $w_i = \frac{r}{x-c_i}$. For each subset J of I let R_J be the ring of all formal power series

$$a_0 + \sum_{j \in J} \sum_{n=1}^{\infty} a_{jn} w_j^n,$$

where $a_0, a_{jn} \in \hat{K}$ and $|a_{jn}|$ tends to 0 as $n \to \infty$ for all $j \in J$. Then R_J is a complete normed domain. Moreover, R_J is a principal domain and every ideal of R_J is generated by an element of $K[w_i]$ for each $i \in J$. Set $P = \mathrm{Quot}(R_I)$, $P_i = \mathrm{Quot}(R_{I \smallsetminus \{i\}})$, and $P_i' = \mathrm{Quot}(R_{\{i\}})$, for $i \in I$. Then P and the P_i, P_i''s satisfy the requirements of 2.

It is now possible to choose for each $i \in i$ a Galois extension F_i of F in P_i' with Galois group G_i. Moreover, if E is a finite extension of a field E_0 with Galois group Γ acting on G, it is possible to choose c_i and F_i such that Γ acts on the whole date in a compatible way. Now it is possible to apply the conclusion of 2.

Literature: [Jar10, Chapters, 2,3,4]

4. Arno Fehm: Non-ample Fields.

By definition, finite fields are non-ample. Using Riemann-Roch it is possible to prove that every function field of one variable over a field K is non-ample. Moreover, if $\mathrm{trans.deg}(F/K) = 1$ and F is a compositum of function fields of one variable over K of bounded genus, then F is non-ample. To prove that every number field is non-ample, one applies a theorem of Faltings. In order to give examples of infinite non-ample extensions of $\mathbb{Q}$ one proves that if A is an Abelian variety defined over an ample field K of characteristic 0, then $\dim_{\mathbb{Q}}(A(K) \otimes \mathbb{Q}) = \infty$. The proof applies the Mordell-Lang Conjecture proved by Faltings and others. Finally, using the "gonality" of a curve (and its properties developed by Poonen) and a theorem of Frey (also relying on the Mordell-Lang Conjecture), one may follow Corvaja and construct for each positive integer d a sequence of linearly disjoint sequence of Abelian extensions of $\mathbb{Q}$ of degree d whose compositum is non-ample.

Literature: [Jar10, Chapter 6]

References

[Jar10] M. Jarden, *Algebraic Patching*, manuscript containing 426 pages, in preparation.