Introduction to fundamental groups with a view towards...

Preliminary lectures notes for the G.A.M.S.C. summer school
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Contents

Forewords

In its current form, the text below should not be regarded as very well-structured lecture notes but rather as a diary of my own learning about fundamental groups.

Section 1 is the result of a master 2 thesis I directed about chapter V of [?]. Thus it is rather detailled and, hopefully, fills some of the gaps in the litterature.

Section 2 contains some of the most striking results about etale fundamental groups and corresponds to my reading of chapter 8 to 12 of [?]. I tried and sketch most of the proofs there, insisting on the main arguments. I hope the final result will be more readable and synthetic than the original source.

Section 3 is just an overview of some classical results about anabelian geometry. I have not included any proofs (nor even sketch them) for such works already exist in the litterature and are, in my opinion, very accessible to beginners.

Notation

- For any field k and any algebraically field closed extension $k \hookrightarrow \Omega$:
  – $k \hookrightarrow k^s$: separable closure of k in Ω;
  – $k \hookrightarrow k^i$: inseparable closure of k in Ω;
  – $k \hookrightarrow \overline{k}$: algebraic closure of k in Ω;
  – $\Gamma_k = \mathrm{Gal}(k^s|k)$ absolute Galois group of k.

- For any integers $g, r \geq 0$, $\Gamma_{g,r}$ is the group defined by the generators $a_1, \ldots, a_g, b_1, \ldots, b_g, \gamma_1, \ldots, \gamma_r$ with the single relation $[a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_r = 1$.

- Given a category $\mathcal{C}$ and two objects X, Y in $\mathcal{C}$, we will write $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\mathrm{Mono}_{\mathcal{C}}(X, Y)$, $\mathrm{Epi}_{\mathcal{C}}(X, Y)$, $\mathrm{StrictEpi}_{\mathcal{C}}(X, Y)$, $\mathrm{Isom}_{\mathcal{C}}(X, Y)$ for the morphisms, monomorphisms, epimorphisms, strict epimorphisms and isomorphisms from X to Y in $\mathcal{C}$ respectively.

1 Galois categories

1.1 Galois categories

Définition 1.1 *A **Galois category** is a category $\mathcal{C}$ such that there exists a covariant functor $F : \mathcal{C} \to FSets$ satisfying the following axioms:*

1. *Finite projective limits exist in $\mathcal{C}$ (or, equivalently, $\mathcal{C}$ has a final object $e_{\mathcal{C}}$ and finite fiber products exist in $\mathcal{C}$).*

2. *Finite inductive limits exist in $\mathcal{C}$ (or, equivalently, finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$). In particular, there is an initial object $\emptyset_{\mathcal{C}}$ in $\mathcal{C}$.*

3. *Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \stackrel{u'}{\to} X' \stackrel{u''}{\to} X$, where u' is a strict epimorphism[1] and u'' is a monomorphism which is an isomorphism onto a direct factor of X.[2]*

4. *F is left exact (in particular F commutes with finite projective limits).*

5. *F is right exact (in particular F commutes with finite coproducts and categorical quotients by finite groups of automorphisms and sends strict epimorphisms to strict epimorphisms).*

6. *For any morphism $u : Y \to X$ in $\mathcal{C}$, $F(u)$ is an isomorphism if and only if u is an isomorphism.*

Given a Galois category $\mathcal{C}$, a functor $F : \mathcal{C} \to FSets$ satisfying axioms (4), (5), (6) is called a *fibre functor for $\mathcal{C}$*. To any fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$ is associated the *fundamental group of $\mathcal{C}$ with base point F*:

$$\pi_1(\mathcal{C}; F) := \mathrm{Aut}_{Fct}(F).$$

Also, to any two fibre functors $F_i : \mathcal{C} \to FSets$ for $\mathcal{C}$, $i = 1,\ 2$ is associated the *set of paths from F_1 to F_2 in $\mathcal{C}$*:

$$\pi_1(\mathcal{C}; F_1, F_2) := \mathrm{Isom}_{Fct}(F_1, F_2).$$

Example 1.2

1. For any field k, let FSA_k denote the category of finite separable k-algebras. Then FSA_k is separable with fibre functors:

$$F_i := \mathrm{Hom}_{FSA_k}(-, \Omega),$$

where $i : k \hookrightarrow \Omega$ is an algebraically closed field extension. In that case:

$$\pi_1(FSA_k; F_i) = \Gamma_k.$$

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[1]Recall that a morphism $u : X \to Y$ in $\mathcal{C}$ is a strict epimorphism if for any object Z in $\mathcal{C}$, the map $u\circ : \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$ is injective and induces a bijection onto the set of all morphism $v : X \to Z$ in $\mathcal{C}$ such that $f \circ p_1 = f \circ p_2$, where $p_i : X \times_Y X \to X$ denotes the ith projection, $i = 1,\ 2$.

[2]And, in that case, the decomposition $Y \stackrel{u'}{\to} X' \stackrel{u''}{\to} X$ is unique in the sense that for any two such decompositions $Y \stackrel{u'_i}{\to} X'_i \stackrel{u''_i}{\to} X = X'_i \coprod X''_i$, $i = 1,\ 2$ there exists an isomorphism $\omega : X'_1 \tilde{\to} X'_2$ such that $\omega \circ u'_1 = u'_2$ and $u''_2 \circ \omega = u''_1$.

2. For any connected, locally arcwise connected and localy simply connected topological space B, let FR_B^{top} denote the category of finite topological covers of B. Then FR_B^{top} is Galois with fibre functors:

$$F_b: \begin{array}{ccc} FR_B^{top} & \to & FSets \\ f: X \to B & \to & f^{-1}(b) \end{array}, \; b \in B.$$

In that case:

$$\pi_1(FR_B^{top}; F_b) = \widehat{\pi_1^{top}(B, b)}$$

(the profinite completion of the topological fundamental group of B with base point b).

3. For any profinite group Π, let $\mathcal{C}(\Pi)$ denote the category of finite (discrete) sets with continuous Π-action. Then $\mathcal{C}(\Pi)$ is Galois with fibre functor the forgetful functor $For : \mathcal{C}(\Pi) \to FSets$. And, in that case:

$$\pi_1(\mathcal{C}(\Pi); For) = \Pi.$$

Example 1.2 (3) is actually the typical example of Galois categories. Indeed, the fundamental group $\pi_1(\mathcal{C}, F)$ is equipped with a natural structure of profinite group. A basis of open subgroups for the profinite topology is given by the kernels K_X of the evaluation morphisms $\pi_1(\mathcal{C}, F) \to \mathrm{Aut}_{FSets}(F(X))$, $\theta \mapsto \theta(X)$ for objects X in $\mathcal{C}$.

By definition of this topology, a fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$ factors as:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "F"'] & FSets \\ \mathcal{C}(\pi_1(\mathcal{C}, F)) \arrow[ur, "For"'] & \end{tikzcd}$$

Theorem 1.3 *Let $\mathcal{C}$ be a Galois category. Then:*

1. *Any fibre functor $F : \mathcal{C} \to FSets$ induces an equivalence of categories $F : \mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}, F))$.*

2. *For any fibre functors F_1, $F_2 : \mathcal{C} \to FSets$, $\pi_1(\mathcal{C}; F_1, F_2) \neq \emptyset$ and the profinite groups $\pi_1(\mathcal{C}, F_1)$ and $\pi_1(\mathcal{C}, F_2)$ are isomorphic canonically up to inner automorphisms.*

1.2 Proof of the main theorem

Let $\mathcal{C}$ be a Galois category and let $F : \mathcal{C} \to FSets$ be a fibre functor for $\mathcal{C}$.

1.2.1 Categorical lemmas

1.2.2 A few categorical lemmas

We gather here a few elementary categorical lemmas, which will be used below.

Lemma 1.4 *Let $\mathcal{C}$ be a category which admits fibre products and $u : X \to Y$ be a morphism in $\mathcal{C}$.*
(1) $u : X \to Y$ is a monomorphism if and only if the first projection $p_1 : X \times_Y X \to Y$ is an isomorphism.
(2) If $u : X \to Y$ is both a monomorphism and a strict epimorphism then $u : X \to Y$ is an isomorphism.

Proof (1) Observe first that, by definition, $p_1 \circ \Delta_{X|Y} = Id_X$ so, if $p_1 : X \times_Y X \to Y$ is an isomorphism, its inverse is automatically $\Delta_{X|Y} : X \to X \times_Y X$. Assume first that $u : X \to Y$ is a monomorphism. Then, from $p_1 \circ u = p_2 \circ u$, one deduces that $p_1 = p_2$. But, then, $p_1 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1$ and $p_2 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1 = p_2$ so, from the uniqueness in the universal property of the fiber product, one gets $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$. Conversely, assume that $p_1 : X \times_Y X \tilde{\to} Y$ is an isomorphism. Then, for any morphisms $f, g : W \to X$ in $\mathcal{C}$ such that $u \circ f = u \circ g$ there exists a unique morphism $(f, g) : W \to X \times_Y X$ such that $p_1 \circ (f, g) = f$ and $p_2 \circ (f, g) = g$. From the former equality, one obtains that $(f, g) = \Delta_{X|Y} \circ f$ and, from the latter, that $g = p_2 \circ (f, g) = p_2 \circ \Delta_{X|Y} \circ f = f$.

(2) Since $u : X \to Y$ is a strict epimorphism, the map $u\circ : \mathrm{hom}_{\mathcal{C}}(Y, X) \to \mathrm{hom}_{\mathcal{C}}(Y, Y)$ induces a bijection onto the set of all morphisms $v : Y \to Y$ such that $v \circ p_1 = v \circ p_2$, where $p_i : X \times_Y X \to Y$ is the ith projection, $i = 1,\ 2$. But since $u : X \to Y$ is also a monomorphism, the first projection $p_1 : X \times_Y X \tilde{\to} X$ is an isomorphism with inverse $\Delta_{X|Y} : X \to X \times_Y X$. So $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$, which yields:

$$\begin{aligned} p_2 \circ \Delta_{X|Y} \circ p_1 &= p_2 \\ &= Id_X \circ p_1 \quad = p_1. \end{aligned}$$

Thus $p_1 = p_2$ and, using again that $u : X \to Y$ is a strict epimorphism, we get that there exists $v : Y \to X$ such that $u \circ v = Id_Y$. But, then, $u \circ v \circ u = u = u \circ Id_X$ and, as u is a monomorphism $v \circ u = Id_X$. □

Lemma 1.5 *A Galois category $\mathcal{C}$ is artinian.*

Proof. Let

$$\cdots \overset{t_{n+1}}{\hookrightarrow} T_n \overset{t_n}{\hookrightarrow} \cdots \overset{t_2}{\hookrightarrow} T_1 \overset{t_1}{\hookrightarrow} T_0$$

be a decreasing sequence of monomorphisms in $\mathcal{C}$. By axiom (1), $T_{n+1} \overset{t_{n+1}}{\hookrightarrow} T_n$ is a monomorphism if and only if the first projection $pr_1 : T_{n+1} \times_{T_n} T_{n+1} \tilde{\to} T_{n+1}$ is an isomorphism, which implies, by axiom (4), that the first projection $pr_1 : F(T_{n+1}) \times_{F(T_n)} F(T_{n+1})) \tilde{\to} F(T_{n+1})$ is also an isomorphism or, equivalently, that $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is a monomorphism. But since $F(T_0)$ is finite, $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is actually an isomorphism for $n \gg 0$ hence, by axiom (6), $t_{n+1} : T_{n+1} \hookrightarrow T_n$ is also an isomorphism for $n \gg 0$. □

Lemma 1.6 *Let $\mathcal{C}$ be a Galois category with fiber functor F. Then, for any $X_0 \in \mathcal{C}$, $F(X_0) = \emptyset$ if and only if $X_0 = \emptyset_{\mathcal{C}}$.*

Proof. By definition of an initial object, for any $X \in \mathcal{C}$ $|\mathrm{Hom}_{\mathcal{C}}(\emptyset_{\mathcal{C}}, X)| = 1$ so, we denote by $u_X : \emptyset_{\mathcal{C}} \to X$ the unique morphism from $\emptyset_{\mathcal{C}}$ to X in $\mathcal{C}$.

$\Rightarrow$ $F(u_{X_0}) \in \mathrm{Hom}_{Fsets}(F(\emptyset_{\mathcal{C}}), F(X_0)) = \mathrm{Hom}_{Fsets}(F(\emptyset_{\mathcal{C}}), \emptyset)$. But, for any $E \in FSets$, $\mathrm{Hom}_{Fsets}(E, \emptyset) \neq \emptyset$ if and only if $F = \emptyset$. Whence $F(\emptyset_{\mathcal{C}}) = \emptyset$. but, then, $F(u_{X_0}) = Id_{\emptyset}$ is an isomorphism hence, by axiom (6) so is u_{X_0}.

$\Leftarrow$ for any object $X \in \mathcal{C}$, one has a canonical isomorphism $(u_X, Id_X) : \emptyset_{\mathcal{C}} \coprod X \tilde{\to} X$ (with inverse the canonical morphism $i_X : X \tilde{\to} \emptyset_{\mathcal{C}} \coprod X)$ thus $F((u_X, Id_X)) : F(\emptyset_{\mathcal{C}} \coprod X) \tilde{\to} F(X)$ is again an isomorphism. But, by axiom (5) $F(\emptyset_{\mathcal{C}} \coprod X) \simeq F(\emptyset_{\mathcal{C}}) \coprod F(X)$, which forces $|F(\emptyset_{\mathcal{C}})| = 0$ hence $F(\emptyset_{\mathcal{C}}) = \emptyset$. □

1.2.3 Strict pro-representability of $F : \mathcal{C} \to FSets$

The category $Pro(\mathcal{C})$ associated with $\mathcal{C}$ is defined by:
- Objects: projective systems $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j \in I,\ i \geq j}$ in $\mathcal{C}$.

- Morphisms from $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\, i\geq j}$ to $\underline{X}' = (\phi'_{i,j} : X'_i \to X'_j)_{i,j\in I',\, i\geq j}$:

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, \underline{X}') := \varprojlim_{i'\in I'} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, X'_{i'}).$$

Note that $\mathcal{C}$ can be regarded canonically as a full subcategory of $Pro(\mathcal{C})$ and that $F : \mathcal{C} \to FSets$ canonically extends to a functor $Pro(F) : Pro(\mathcal{C}) \to Pro(FSets)$.

The functor $F : \mathcal{C} \to FSets$ is *pro-representable in* $\mathcal{C}$ if there exists $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\, i\geq j} \in Pro(\mathcal{C})$ and a functor isomorphism:

$$\theta : \mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, -)|_{\mathcal{C}} \tilde{\to} F$$

and the functor $F : \mathcal{C} \to FSets$ is *strictly pro-representable in* $\mathcal{C}$ if it is representable and if, in addition, the transition morphisms $\phi_{i,j} : X_i \twoheadrightarrow X_j$ are epimorphisms, $i, j \in I,\ i \geq j$.

Proposition 1.7 *The fibre functor $F : \mathcal{C} \to FSets$ is strictly pro-representable in $\mathcal{C}$ by the projective system $\underline{X} = (\phi_{m,n} : X_m \twoheadrightarrow X_n)_{m,n\in\mathcal{M},\, m\geq n}$ of connected objects in $\mathcal{C}$.*

Proof. The *pointed category associated with* $\mathcal{C}$ is the category $\mathcal{C}^{pt}$ defined by:
- Objects: pairs (X, ζ) with $X \in \mathcal{C}$ and $\zeta \in F(X)$.
- Morphisms from (X_1, ζ_1) to (X_2, ζ_2):

$$\mathrm{Hom}_{\mathcal{C}^{pt}}((X_1, \zeta_1), (X_2, \zeta_2)) = \{u : X_1 \to X_2 \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \mid F(u)(\zeta_1) = \zeta_2\}.$$

Let $\mathcal{M} =$ "$\{(X_m, \zeta_m)\}_{m\in\mathcal{M}}$" denote the set of objects (X, ζ) in $\mathcal{C}^{pt}$ with X connected in $\mathcal{C}$. We are going to show that $\mathcal{M}$ is canonically equipped with a structure of projective system $\mathcal{M} = (\phi_{m,n} : (X_m, \zeta_m) \twoheadrightarrow (X_n, \zeta_n))_{m,n\in\mathcal{M},\, m\geq n}$ the transition morphisms of which are strict epimorphisms. This will rely on the following:

<u>Properties of connected objects:</u>

1. *For any $X_0 \in \mathcal{C}$, $X_0 \in \mathcal{C}$ is connected if and only if for any $\zeta_0 \in F(X_0)$ and $(X, \zeta) \in \mathcal{C}^{pt}$*

$$\mathrm{Mono}_{\mathcal{C}^{pt}}((X, \zeta), (X_0, \zeta_0)) = \mathrm{Isom}_{\mathcal{C}^{pt}}((X, \zeta), (X_0, \zeta_0))$$

   (that is (X_0, ζ_0) is minimal in $\mathcal{C}^{pt}$).

2. *For any connected object $X_0 \in \mathcal{C}$ and for any $X \in \mathcal{C}$,*
   (i) $|\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X, \zeta))| \leq 1$*;*
   (ii) $\mathrm{Hom}_{\mathcal{C}}(X, X_0) = \mathrm{StrictEpi}_{\mathcal{C}}(X, X_0)$.

3. *For any $(X_i, \zeta_i) \in \mathcal{C}^{pt}$, $i = 1, 2$ there exists $(X_0, \zeta_0) \in \mathcal{M}$ such that*

$$\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X_i, \zeta_i)) \neq \emptyset,\ i = 1, 2.$$

Thus $\mathcal{M}$ will define a canonical functor morphism $\theta : \mathrm{Hom}_{Pro(\mathcal{C})}(\mathcal{M}, -)|_{\mathcal{C}} \to F$ by

$$\begin{array}{llll} \theta(X) : & \mathrm{Hom}_{Pro(\mathcal{C})}(\mathcal{M}, X) & \to & F(X) \\ & \underline{u} = (u_m : X_m \to X)_{m\in\mathcal{M}} & \mapsto & (F(u_m)(\zeta_m))_{m\in\mathcal{M}} = F(u_m)(\zeta_m),\ m >> 0. \end{array}$$

and it follows from property (3) that $\theta(X)$ is surjective and from property (2) (i) that $\theta(X)$ is injective hence $\theta : \mathrm{Hom}_{Pro(\mathcal{C})}(\mathcal{M}, -)|_{\mathcal{C}} \tilde{\rightarrow} F$ is, actually, a functor isomorphism as claimed.

It remains to prove properties (1), (2), (3) of connected objects.

1. $\Rightarrow$ Write $X_0 = X_0' \coprod X_0''$. By axiom (5), $\zeta_0 \in F(X_0) = F(X_0') \coprod F(X_0'')$. Assume for instance that $\zeta_0 \in F(X_0')$. Then the canonical inclusion $i : X_0' \hookrightarrow X_0$ in $\mathcal{C}$ induces a monomorphism in $\mathcal{C}^{pt}$ $i : (X_0', \zeta_0) \hookrightarrow (X_0, \zeta_0)$. Since $i : X_0' \hookrightarrow X_0$ is a monomorphism and (X_0, ζ_0) is minimal, $i : X_0' \hookrightarrow X_0$ is necessarily an isomorphism hence $X_0'' = \emptyset$.

   $\Leftarrow$ For any $\zeta_0 \in F(X_0)$ and any $(X, \zeta) \in \mathcal{C}^{pt}$ let $i : (X, \zeta) \hookrightarrow (X_0, \zeta_0)$ be a monomorphism in $\mathcal{C}$. Then, by axiom (3), $i : X \hookrightarrow X_0$ factors as $X \xrightarrow{i'} X_0' \xrightarrow{i''} X_0 = X_0' \coprod X_0''$ with $i' : X_0 \rightarrow X_0'$ a strict epimorphism and $i'' : X_0' \rightarrow X_0$ a monomorphism inducing an isomorphism onto X_0'. Since X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. But $X_0' = \emptyset_{\mathcal{C}}$ is impossible since $X_0 \neq \emptyset_{\mathcal{C}}$ $(\zeta_0 \in F(X_0))$. Hence $X_0'' = \emptyset_{\mathcal{C}}$ and $i'' : X \hookrightarrow X_0$ is an isomorphism. But, then, $i : X \hookrightarrow X_0$ is both a monomorphism and a strict epimorphism hence an isomorphism.

2. (i) For any morphisms $u_i : (X_0, \zeta_0) \rightarrow (X, \zeta)$ in $\mathcal{C}^{pt}$, $i = 1, 2$, one has an exact sequence:

$$\mathrm{Ker}(u_1, u_2) \overset{i}{\hookrightarrow} X_0 \rightrightarrows X.$$

   From axiom (4),

$$F(\mathrm{Ker}(u_1, u_2)) \overset{F(i)}{\hookrightarrow} F(X_0) \rightrightarrows F(X)$$

   is again exact. Thus $\zeta_0 \in \mathrm{Ker}(F(u_1), F(u_2)) = F(\mathrm{Ker}(u_1, u_2))$ and $F(i)(\zeta_0) = \zeta_0$. It follows then from the minimality of (X_0, ζ_0) that $i : \mathrm{Ker}(u_1, u_2) \hookrightarrow X_0$ is actually an isomorphism that is, $u_1 = u_2$.
   (ii) By axiom (3), $u : X \rightarrow X_0$ factors as $X \xrightarrow{u'} X_0' \xrightarrow{u''} X_0' \coprod X_0'' = X_0$, where u' is a strict epimorphism and u'' is a monomorphism inducing an isomorphism onto X_0'. Furthermore, $F(u)(\zeta) = F(u'')(F(u')(\zeta)) = \zeta_0$ thus, by minimality of (X_0, ζ_0), $u'' : X_0' \hookrightarrow X_0$ is actually an isomorphism.

3. Take $X_0 := X_1 \times X_2$, $\zeta_0 := (\zeta_1, \zeta_2) \in F(X_1) \times F(X_2) = F(X_1 \times X_2)$ (by axiom (4)) and $u_i := pr_i : X_0 \rightarrow X_i$ the ith projection, $i = 1, 2$. Then $\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X_i, \zeta_i)) \neq \emptyset$, $i = 1, 2$. So, it is enough to prove that for any $(X, \zeta) \in \mathcal{C}^{pt}$ there exists $(X_0, \zeta_0) \in \mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X, \zeta)) \neq \emptyset$.
   If $(X, \zeta) \in \mathcal{M}$ then $Id : (X, \zeta) \rightarrow (X, \zeta)$ works. Else, there exists $(X_1, \zeta_1) \in \mathcal{C}^{pt}$ and a monomorphism $u_1 : (X_1, \zeta_1) \hookrightarrow (X, \zeta)$ which is not an isomorphism in $\mathcal{C}^{pt}$. If the claim were not true, one could construct inductively an infinite sequence

$$(X_{n+1}, \zeta_{n+1}) \overset{u_{n+1}}{\hookrightarrow} (X_n, \zeta_n) \overset{u_n}{\hookrightarrow} \ldots \overset{u_2}{\hookrightarrow} (X_1, \zeta_1) \overset{u_1}{\hookrightarrow} (X, \zeta),$$

   with $u_n : X_n \hookrightarrow X_{n-1}$ a monomorphism which is not an isomorphis in $\mathcal{C}$. But this would contradict lemma 1.5. $\square$

Lemma 1.8 *For any connected object X_0 in $\mathcal{C}$:*
(1) $\mathrm{Hom}_{\mathcal{C}}(X_0, X_0) = \mathrm{Aut}_{\mathcal{C}}(X_0)$*;*
(2) For any $\zeta_0 \in F(X_0)$, the evaluation map:

$$\begin{array}{rccc} ev_{\zeta_0}: & \mathrm{Aut}_{\mathcal{C}}(X_0) & \hookrightarrow & F(X_0) \\ & u : X_0 \tilde{\to} X_0 & \mapsto & F(u)(\zeta_0) \end{array}$$

is injective.
(3) For any morphism $u : X_0 \to X$ in $\mathcal{C}$, if $u : X_0 \twoheadrightarrow X$ is a strict epimorphism then X is also connected.

Proof.

1. By axiom (3), any morphism $u : X_0 \to X_0$ in $\mathcal{C}$ factors as $X_0 \stackrel{u'}{\to} X'_0 \stackrel{u''}{\to} X_0 = X'_0 \coprod X''_0$ with $u' : X_0 \to X'_0$ a strict epimorphism and $u'' : X'_0 \to X_0$ a monomorphism inducing an isomorphism onto X'_0. But since X_0 is connected either $X'_0 = \emptyset_{\mathcal{C}}$ or $X''_0 = \emptyset_{\mathcal{C}}$. The former implies $X_0 = \emptyset_{\mathcal{C}}$ and then the claim is straightforward. The latter implies $X_0 = X''_0$ thus $u'' : X'_0 \to X_0$ is an isomorphism and $u : X_0 \to X_0$ is a strict epimorphism. Hence $F(u) : F(X_0) \twoheadrightarrow F(X_0)$ is surjective hence bijective since $F(X_0)$ is finite; The conclusion then follows from axiom (6).

2. For any automorphisms $u_i : X_0 \tilde{\to} X_0$ in $\mathcal{C}$, $i = 1,\ 2$ such that $F(u_1)(\zeta_0) = F(u_2)(\zeta_0) = \zeta$, $u_i : (X_0, \zeta_0) \to (X, \zeta)$ is a morphism in $\mathcal{C}^{pt}$, $i = 1,\ 2$ hence, by property (2) (i) of the connected objects X_0, $u_1 = u_2$.

3. If not, there would exist a decomposition $X = X' \coprod X''$ in $\mathcal{C}$ with $X', X'' \neq \emptyset_{\mathcal{C}}$. Fix $\zeta_0 \in F(X_0)$. Then, by axiom (5), $F(u)(\zeta_0) \in F(X) = F(X') \coprod F(X'')$. Assume, for instance, that $\zeta' = F(u)(\zeta_0) \in F(X')$. Then, from property (3) of connected objects, there exist $(X'_0, \zeta'_0) \in \mathcal{M}$ and a morphism $v : (X'_0, \zeta'_0) \to (X_0 \times X', (\zeta_0, \zeta'))$ in $\mathcal{C}^{pt}$. Then $w := p_1 \circ v : (X'_0, \zeta'_0) \to (X_0, \zeta_0)$ and $w' := p_2 \circ v : (X'_0, \zeta'_0) \to (X', \zeta')$ are morphisms in $\mathcal{C}^{pt}$. But from property (2) (ii) of connected object $w : X'_0 \to X_0$ is automatically a strict epimorphism, so is $u \circ w : X'_0 \to X$. Since $F(u \circ w)(\zeta_0) = \zeta' = F(w')(\zeta_0)$, it follows from property (2) (i) of the connected object X_0 that $u \circ w = w'$, which contradicts $X'' \neq \emptyset_{\mathcal{C}}$. $\square$

Remark 1.9 For any $X \in \mathcal{C}$, write $F(X) = \{\zeta_1, \dots, \zeta_n\}$. Then, from property (3) of connected objects, there exists $(X_0, \zeta_0) \in \mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X, \zeta_i)) \neq \emptyset$, $i = 1, \dots, n$. Thus the canonical evalutation map $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \hookrightarrow F(X)$, $u : X_0 \to X \mapsto F(u)(\zeta_0)$ is surjective. But, from property (2) (i) of connected objects, it is also injective, hence bijective.

A connected object X_0 in $\mathcal{C}$ is *Galois in $\mathcal{C}$* if for any $\zeta_0 \in F(X_0)$ the evaluation map $ev_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$, $u : X_0 \tilde{\to} X_0 \mapsto F(u)(\zeta_0)$ is bijective. By lemma 1.8 (2), this is equivalent to one of the following:

1. $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$;

2. $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$;

3. $|\mathrm{Aut}_{\mathcal{C}}(X_0)| = |F(X_0)|$.

Denote by $\mathcal{G} \subset \mathcal{M}$ the subset of all $(X_0, \zeta_0) \in \mathcal{M}$ with X_0 Galois.

Proposition 1.10 *For any $(X, \zeta) \in \mathcal{C}^{pt}$ there exists $(X_0, \zeta_0) \in \mathcal{G}$ such that $\mathrm{Hom}_{\mathcal{C}^{pt}}((X_0, \zeta_0), (X, \zeta)) \neq \emptyset$. (In other words, $\mathcal{G}$ is cofinal in $\mathcal{M}$). In particular, the fibre functor $F : \mathcal{C} \to FSets$ is strictly pro-representable in $\mathcal{C}$ by the projective system $\underline{X} = (\phi_{g,h} : X_g \twoheadrightarrow X_h)_{g,h \in \mathcal{G},\, g \geq h}$ of Galois objects in $\mathcal{C}$.*

Proof. Fix first $(X_0, \zeta_0) \in \mathcal{M}$ such that the canonical evaluation map $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \tilde{\to} F(X),\ u : X_0 \to X \mapsto F(u)(\zeta_0)$ is bijective. Write $\mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{u_1, \dots, u_n\}$, $\zeta_i := F(u_i)(\zeta_0)$ and $pr_i : X^n \to X$ for the ith projection. Consider the diagonal morphism:

$$\begin{array}{cccc} \pi : & X_0 & \to & X^n \\ & x_0 & \mapsto & (u_1(x_0), \dots, u_n(x_0)) \end{array}.$$

Then, by definition, $pr_i \circ \pi = u_i$, $i = 1, \dots, n$.

By axiom (3), $\pi : X_0 \to X^n$ factors as $X_0 \stackrel{\pi'}{\to} G' \stackrel{\pi''}{\to} X^n = G' \coprod G''$ with π' a strict epimorphism in $\mathcal{C}$ and π'' a monomorphism inducing an isomorphism onto the direct factor G' of X^n in $\mathcal{C}$. We claim that G' is Galois. The conclusion will then follow from the fact that $pr_i \circ \pi'' : (G', \zeta_0) \to (X, \zeta_i)$ is a morphism in $\mathcal{C}^{pt}$, $i = 1, \dots, n$.

From lemma 1.8 (3), G' is connected in $\mathcal{C}$. Set $\gamma'_0 := F(\pi')(\zeta_0) \in F(G')$. Then we are to prove that the canonical evaluation map $ev_{\gamma'_0} : \mathrm{Aut}_{\mathcal{C}}(G') \to F(G')$, $\omega : G' \tilde{\to} G' \mapsto F(\omega)(\gamma'_0)$ is surjective or, in other words, for any $\gamma' \in F(G')$, we are to find an automorphism $\omega : G' \tilde{\to} G'$ such that $F(\omega)(\gamma'_0) = \gamma'$. From property (3) of connected objects, there exists $(\tilde{X}_0, \tilde{\zeta}_0) \in \mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{C}^{pt}}((\tilde{X}_0, \tilde{\zeta}_0), (X_0, \zeta_0)) \neq \emptyset$ and $\mathrm{Hom}_{\mathcal{C}^{pt}}((\tilde{X}_0, \tilde{\zeta}_0), (G', \gamma')) \neq \emptyset$. So, up to replacing (X_0, ζ_0) with $(\tilde{X}_0, \tilde{\zeta}_0)$, we may also assume that there exists a morphism $\rho : (X_0, \zeta_0) \to (G', \gamma')$ in $\mathcal{C}^{pt}$. But, on the one hand $F(\omega \circ \pi')(\zeta_0) = F(\omega)(\gamma'_0)$ and, on the other hand, $\gamma' = F(\rho)(\zeta_0)$. Thus, by property (2) (i) of the connected object X_0, $F(\omega)(\gamma'_0) = \gamma'$ if and only if $\omega \circ \pi' = \rho$.

By construction, $F(u_i)(\zeta_0) \neq F(u_j)(\zeta_0)$, $1 \leq i \neq j \leq n$ hence, by property (2) (i) of the connected object X_0, $u_i \neq u_j$, $1 \leq i \neq j \leq n$. Since $\pi' : X_0 \to G'$ is a strict epimorphism, we thus have $pr_i \circ \pi'' \neq pr_j \circ \pi''$, $1 \leq i \neq j \leq n$. But since X_0 is connected, $\rho : X_0 \to G'$ is automatically a strict epimorphism hence $pr_i \circ \pi'' \circ \rho \neq pr_j \circ \pi'' \circ \rho$, $1 \leq i \neq j \leq n$. Eventually, since G' is connected $F(pr_i \circ \pi'' \circ \rho)(\zeta_0) \neq F(pr_j \circ \pi'' \circ \rho)(\zeta_0)$, $1 \leq i \neq j \leq n$, whence $\{F(u_i)(\zeta_0)\}_{1 \leq i \leq n} = F(X) = \{F(pr_i \circ \pi'' \circ \rho)(\zeta_0)\}_{1 \leq i \leq n}$. From which it follows that there exists a permutation $\sigma \in \mathcal{S}_n$ such that $F(pr_i \circ \pi'' \circ \rho)(\zeta_0) = F(pr_{\sigma(i)} \circ \pi'' \circ \pi')(\zeta_0)$ and, in turn, σ defines an isomorphism $\sigma : X^n \tilde{\to} X^n$ (by permuting coordinates) such that $\sigma \circ \pi'' \circ \pi' = \pi'' \circ \rho$. But, then, from the unicity of the decomposition in axiom (3), there exists an automorphism $\omega : G' \tilde{\to} G'$ satisfying $\sigma \circ \pi'' = \pi'' \circ \omega$ and $\omega \circ \pi' = \rho$. $\square$

Remark 1.11 Actually, for any object $X \in \mathcal{C}$ there exists $(X_0, \zeta_0) \in \mathcal{G}$ such that the canonical evaluation morphism $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \tilde{\to} F(X)$, $u : X \to X_0 \mapsto F(u)(\zeta_0)$ is bijective.

Corresponding to the projective system $(\phi_{g,h} : X_g \twoheadrightarrow X_h)_{g,h \in \mathcal{G},\, g \geq h}$ in $Pro(\mathcal{C})$, one has a projective system $(F(\phi_{g,h}) : F(X_g) \twoheadrightarrow F(X_h))_{g,h \in \mathcal{G},\, g \geq h}$ in $Pro(FSets)$.

For any $g \in \mathcal{G}$, set $G_g := \mathrm{Aut}_{\mathcal{C}}(X_g)$.Then for any $g,\ h \in \mathcal{G},\ g \geq h$, the map

$$\psi_{g,h} : G_g \stackrel{ev_{\zeta_g}}{\tilde{\to}} F(X_g) \stackrel{F(\phi_{g,h})}{\twoheadrightarrow} F(X_h) \stackrel{ev_{\zeta_h}^{-1}}{\tilde{\to}} G_h$$

is the unique map $\psi_{g,h} : G_g \to G_h$ making the following diagrams commute:

$$\begin{array}{ccc} X_g & \xrightarrow{u} & X_g \ , \ u \in G_g \\ {\scriptstyle \phi_{g,h}} \downarrow & & \downarrow {\scriptstyle \phi_{g,h}} \\ X_h & \xrightarrow[\psi_{g,h}(u)]{} & X_h \end{array}$$

and, in particular, is a group epimorphism. This endows the G_g, $g \in \mathcal{G}$ with a structure of projective system $(\psi_{g,h} : G_g \twoheadrightarrow G_h)_{g,\, h \in \mathcal{G},\, g \geq h}$. Set $G := \varprojlim G_g$. With these notation one gets the canonical isomorphism of profinite sets:

$$\begin{array}{rlcl} ev^{(2)}_{\underline{\zeta}} : & G & \tilde{\to} & \varprojlim F(X_g) \\ & (u_g : X_g \tilde{\to} X_g)_{g \in \mathcal{G}} & \mapsto & (F(u_g)(\zeta_g))_{g \in \mathcal{G}}. \end{array}$$

On the other hand, one can then define a map of profinite sets:

$$\begin{array}{rlcl} ev^{(1)}_{\underline{\zeta}} : & \pi_1(\mathcal{C}; F) & \to & \varprojlim F(X_g) \\ & \Theta & \mapsto & (\Theta(X_g)(\zeta_g))_{g \in \mathcal{G}}. \end{array}$$

Lemma 1.12 *The map* $ev^{(1)}_{\underline{\zeta}} : \pi_1(\mathcal{C}; F) \to \varprojlim F(X_g)$ *is an isomorphism of profinite sets.*

Proof. Let $\Theta, \Theta' \in \pi_1(\mathcal{C}; F)$ such that $\Theta(X_g)(\zeta_g) = \Theta'(X_g)(\zeta_g)$, $g \in \mathcal{G}$. Then, for any $X \in \mathcal{C}$, there exists $g \in \mathcal{G}$ such that the canonical evaluation morphism $ev_{\zeta_g} : \mathrm{Hom}_{\mathcal{C}}(X_g, X) \tilde{\to} F(X)$, $u : X \to X_g \mapsto F(u)(\zeta_g)$ is bijective. But $\Theta(X)(F(u)(\zeta_g)) \overset{(*)}{=} F(u)\Theta(X_g)(\zeta_g) \overset{(**)}{=} F(u)\Theta'(X_g)(\zeta_g) \overset{(*)}{=} \Theta'(X)(F(u)(\zeta_g))$, where the equalities $(*)$ are just the definition of a functor morphism and the equality $(**)$ is the assumption that $\Theta(X_g)(\zeta_g) = \Theta'(X_g)(\zeta_g)$. So $\Theta(X) = \Theta'(X)$, whence the injectivity.

For any $(\eta_g)_{g \in \mathcal{G}} \in \varprojlim F(X_g)$ there exists a unique $(u_g)_{g \in \mathcal{G}} \in G$ such that $F(u_g)(\zeta_g) = \eta_g$, $g \in \mathcal{G}$. Then, using the canonical isomorphism $ev_{\zeta_g} : G_g \tilde{\to} F(X_g)$, define $\Theta(X_g)(F(u)(\zeta_g)) := F(u \circ u_g)(\zeta_g)$, $u \in G_g$. Then the maps $\Theta(X_g) : F(X_g) \tilde{\to} F(X_g)$, $g \in \mathcal{G}$ are well-defined isomorphisms and, for any $g, h \in \mathcal{G}$, for any morphism $\phi : X_g \to X_h$ in $\mathcal{C}$, writing $\phi = \alpha \circ \phi_{g,h}$ with $\alpha \in G_h$, one gets:

$$\begin{aligned} \Theta(X_h)F(\alpha \circ \phi_{g,h})(F(u)(\zeta_g)) &= \Theta(X_h)F(\alpha \circ \phi_{g,h} \circ u)(\zeta_g) \\ &= \Theta(X_h)F(\alpha \circ \psi_{g,h}(u) \circ \phi_{g,h}(\zeta_g) \\ &= \Theta(X_h)F(\alpha \circ \psi_{g,h}(u))(\zeta_h) \\ &= F(\alpha \circ \psi_{g,h}(u) \circ u_h)(\zeta_h) \\ &= F(\alpha \circ \psi_{g,h}(u))(\eta_h) \end{aligned}$$

whereas:

$$\begin{aligned} F(\alpha \circ \phi_{g,h}) \circ \Theta(X_h)(F(u)(\zeta_g)) &= F(\alpha \circ \phi_{g,h})(F(u \circ u_g)(\zeta_g)) \\ &= F(\alpha \circ \phi_{g,h} \circ u)(\eta_g) \\ &= F(\alpha \circ \psi_{g,h}(u) \circ \phi_{g,h})(\eta_g) \\ &= F(\alpha \circ \psi_{g,h}(u))(\eta_h), \end{aligned}$$

That is $\Theta(X_h) \circ F(\phi) = F(\phi) \circ \Theta(X_h)$. Using again proposition 1.10, for any $X \in \mathcal{C}$, there exists $g \in \mathcal{G}$ such that the canonical evaluation morphism $ev_{\zeta_g} : \mathrm{Hom}_{\mathcal{C}}(X_g, X) \tilde{\to} F(X)$, $\phi : X \to X_g \mapsto F(\phi)(\zeta_g)$ is bijective. Then, set $\Theta(X)(F(\phi)(\zeta_g)) = F(\phi \circ u_g)$, $\phi : X_g \to X \in \mathrm{Hom}_{\mathcal{C}}(X_g, X)$ and check that this defines an element $\Theta \in \pi_1(\mathcal{C};F)$ such that $ev^{(1)}_{\underline{\zeta}}(\Theta) = (\eta_g)_{g\in\mathcal{G}}$.

Eventually, $ev^{(1)}_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \rightarrow \underleftarrow{\lim} F(X_g)$ is an homeomorphism since $ev^{(1)}_{\underline{\zeta}}(K_{X_g})$ is the inverse image of ζ_g *via* the canonical projection $\underleftarrow{\lim}\, F(X_g) \rightarrow F(X_g)$, $g \in \mathcal{G}$. $\square$

Thus, we have built a canonical isomorphism of profinite sets:

$$c_{\underline{\zeta}} : ev^{(2)\ -1}_{\underline{\zeta}} \circ ev^{(1)}_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \tilde{\rightarrow} G$$

Since for any $\Theta,\ \Theta' \in \pi_1(\mathcal{C};F)$,

$$\begin{aligned} F(c_{\underline{\zeta}}(\Theta')c_{\underline{\zeta}}(\Theta))(\zeta_g) &= F(c_{\underline{\zeta}}(\Theta'))F(c_{\underline{\zeta}}(\Theta))(\zeta_g) \\ &= F(c_{\underline{\zeta}}(\Theta'))\Theta(X_g)(\zeta_g) \\ &= \Theta(X_g)F(c_{\underline{\zeta}}(\Theta'))(\zeta_g) \\ &= \Theta(X_g)\Theta'(X_g)(\zeta_g) \\ &= F(c_{\underline{\zeta}}(\Theta\circ\Theta')(\zeta_g) \end{aligned}$$

$c_{\underline{\zeta}}$ actually induces a profinite group isomorphism:

$$c_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \tilde{\rightarrow} G^{op}.$$

We are going to use this description of $\pi_1(\mathcal{C};F)$ to construct a pseudo-inverse to $F : \mathcal{C} \rightarrow \mathcal{C}(\pi_1(\mathcal{C};F))$.

1.2.4 Pseudo-inverse to $F : \mathcal{C} \rightarrow \mathcal{C}(\pi_1(\mathcal{C};F))$

From now on, write $\Pi := \pi_1(\mathcal{C};F)$ and $\Pi_g := G_g^{op}$, $g \in \mathcal{G}$.

Proposition 1.13 *For any object E in $\mathcal{C}(\Pi)$, there exists an object $G(E)$ in $\mathcal{C}$ and an isomorphism $\gamma_E : E \tilde{\rightarrow} FG(E)$ in $\mathcal{C}(\Pi)$ such that for any object X in $\mathcal{C}$ the map*

$$\begin{array}{rlcl} \omega(X): & \mathrm{Hom}_{\mathcal{C}}(G(E),X) & \rightarrow & \mathrm{Hom}_{\mathcal{C}(\Pi)}(E,F(X)) \\ & u : G(E) \rightarrow X & \mapsto & F(u)\circ\gamma_E : E \rightarrow F(X) \end{array}$$

is bijective. Furthermore, this construction is functorial and defines a pseudo-inverse $G : \mathcal{C}(\Pi) \rightarrow \mathcal{C}$ to $F : \mathcal{C} \rightarrow \mathcal{C}(\Pi)$.

Proof.

1. Definition of $G(E)$ and $\gamma_E : E \tilde{\rightarrow} F(G(E))$. First observe that it is enough to define $G(E)$ for connected objects E in $\mathcal{C}(\Pi)$. Indeed, if

$$E = \coprod_{E_0\in\pi_0(E)} E_0$$

is the decomposition of any object E in $\mathcal{C}(\Pi)$ into connected components then

$$G(E) = \coprod_{E_0\in\pi_0(E)} G(E_0)$$

works by axioms (2) and (5).
So let E be a connected object in $\mathcal{C}(\Pi)$ and fix $\epsilon \in E$. Then there exists $g \in \mathcal{G}$ such that the continuous surjective map $ev_\epsilon : \Pi \twoheadrightarrow E$, $\sigma \mapsto \sigma \cdot \epsilon$ factors through

$$\begin{tikzcd}
\Pi \arrow[r, two heads, "ev_\epsilon"] \arrow[d] & E \\
\Pi_g \arrow[ur, two heads, "ev_\epsilon"'] &
\end{tikzcd}$$

and this induces an isomorphism $\overline{ev}_\epsilon : \Pi_g/\mathrm{Stab}_{\Pi_g}(\epsilon)\tilde{\rightarrow}E$ in $\mathcal{C}(\Pi)$. We set

$$G(E) := X_g/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op}.$$

Then, we have the following canonical isomorphisms in $\mathcal{C}(\Pi)$:

$$\begin{array}{rcll}
F(G(E)) & \tilde{\rightarrow} & F(X_g)/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op} & \text{(by axiom (5))} \\
& \overset{ev_{\zeta_g}^{-1}}{\tilde{\rightarrow}} & G_g/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op} & \\
& \tilde{\rightarrow} & \Pi_g/\mathrm{Stab}_{\Pi_g}(\epsilon) & \\
& \overset{\overline{ev}_\epsilon}{\tilde{\rightarrow}} & E, &
\end{array} \tag{1}$$

which define γ_E^{-1}. The above definitions of $G(E)$ and $\gamma_E : E\tilde{\rightarrow}F(G(E))$ do not depend on $g \in \mathcal{G}$ up to isomorphisms in $\mathcal{C}$. Indeed, for any $g' \in \mathcal{G}$, $g' \geq g$, the canonical morphism $X_{g'}/(\mathrm{Stab}_{\Pi_{g'}}(\epsilon))^{op} \rightarrow X_g/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op}$ in $\mathcal{C}$ induces, by axiom (5) and the above, a bijection $F(X_{g'})/(\mathrm{Stab}_{\Pi_{g'}}(\epsilon))^{op}\tilde{\rightarrow}(E\tilde{\rightarrow})F(X_g)/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op}$ hence, by axiom (6), is already an isomorphism in $\mathcal{C}$.

2. For any object X in $\mathcal{C}$, the map $\omega(X)$ is bijective.

   (a) $\omega(X)$ is injective. Indeed, for any two morphisms $u_1,\ u_2 : G(E) \rightarrow X$ in $\mathcal{C}$ such that

   $$\begin{array}{lll}
   \omega(X)(u_1) & = \gamma_E \circ F(u_1) & \\
   & = \omega(X)(u_2) & = \gamma_E \circ F(u_2)
   \end{array}$$

   $F(u_1) = F(u_2)$ so the canonical map:

   $$\mathrm{Ker}(F(u_1), F(u_2))\tilde{\rightarrow}F(G(E))$$

   is an isomorphism hence, by axioms (5) and (6), so is:

   $$\mathrm{Ker}(u_1, u_2)\tilde{\rightarrow}G(E)$$

   whence $u_1 = u_2 : G(E) \rightrightarrows X$.

   (b) $\omega(X)$ is surjective. Fix $\epsilon \in E$ and for any morphism $\alpha : E \rightarrow F(X)$ in $\mathcal{C}(\Pi)$ set $\zeta := \alpha(\epsilon) \in F(X)$. Then one can always find $g \in \mathcal{G}$ and a morphism $u : (X_g, \zeta_g) \rightarrow (X, \zeta)$ in $\mathcal{C}^{pt}$ such that, in addition, the action of Π on E factors through Π_g. But $u : X_g \to X$ also factors through

$$\begin{tikzcd} X_g \arrow[r, "u"] \arrow[d] & X, \\ X_g/(\mathrm{Stab}_{\Pi_g}(u))^{op} \arrow[ur, "\overline{u}"'] & \end{tikzcd}$$

where $(\mathrm{Stab}_{\Pi_g}(u))^{op} \subset G_g$ is the subgroup of all $\sigma \in G_g$ such that $u \circ \sigma = u$. Since $\alpha : E \to F(X)$ is a morphism in $\mathcal{C}(\Pi)$, for any $\sigma \in \Pi_g$, $\sigma \cdot \epsilon = \epsilon$ implies that $\sigma \cdot \zeta = \zeta$ but $\zeta = F(u)(\zeta_g)$, whence $F(u \circ \sigma)(\zeta_g) = \sigma \cdot \zeta = \zeta = F(u)(\zeta_g)$. Thus, by minimality of (X_g, ζ_g), $u \circ \sigma = u$. So $(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op} \subset (\mathrm{Stab}_{\Pi_i}(u))^{op}$. This yields a canonical morphism

$$u_\alpha : G(E) = X_g/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op} \to X_g/(\mathrm{Stab}_{\Pi_g}(u))^{op} \stackrel{\overline{u}}{\to} X.$$

It remains to check that $F(u_\alpha) \circ \gamma_E = \alpha$. As both $F(u_\alpha) \circ \gamma_E$ ans α are Π_g-equivariant and as Π_g acts transitively on E, it is enough to prove that $F(u_\alpha) \circ \gamma_E(\epsilon) = \alpha(\epsilon) = \zeta = F(u)(\zeta_g)$. Using (1), one has: $\gamma_E(\epsilon) = F(p)(\zeta_g)$, where $p : X_g \to X_g/(\mathrm{Stab}_{\Pi_g}(\epsilon))^{op}$ denotes the quotient morphism in $\mathcal{C}$. As a result:

$$F(u_\alpha) \circ \gamma_E(\epsilon) = F(u_\alpha \circ p)(\zeta_g) = F(u)(\zeta_g).$$

3. Functoriality. For any morphism $\alpha : E \to E'$ in $\mathcal{C}(\Pi)$, $\gamma'_E \circ \alpha : E \to F(G(E'))$ is again a morphism in $\mathcal{C}(\Pi)$ hence, since $\omega(G(E'))$ is bijective, there exist a unique morphism $G(\alpha) : G(E) \to G(E')$ in $\mathcal{C}$ such that $\gamma_{E'} \circ \alpha = G(\alpha) \circ \gamma_E$. Then, for any sequence $E \stackrel{\alpha}{\to} E' \stackrel{\alpha'}{\to} E''$ of morphism in $\mathcal{C}(\Pi)$ one has $G(\alpha' \circ \alpha) \circ \gamma_E = \gamma_{E''} \circ \alpha' \circ \alpha$ and $G(\alpha') \circ G(\alpha) \circ \gamma_E = G(\alpha') \circ \gamma_{E'} \circ \alpha = \gamma_{E''} \circ \alpha' \circ \alpha$, whence, by unicity, $G(\alpha' \circ \alpha) = G(\alpha') \circ G(\alpha)$. That is, $G : \mathcal{C}(\Pi) \to \mathcal{C}$ is a functor. One then checks that $\gamma : Id_{\mathcal{C}(\Pi)} \tilde{\to} F \circ G$ is a functor isomorphism. Similarly, for any object X in $\mathcal{C}$, set $\delta_X := \omega(F(X))^{-1}(Id_{F(X)}) : G(F(X)) \tilde{\to} X$ then, $\delta : G \circ F \tilde{\to} Id_{\mathcal{C}}$ is also a functor isomorphism. Furthermore, it follows from the definitions that for any objects E in $\mathcal{C}(\Pi)$ and X in $\mathcal{C}$ one has:

$$\gamma_{F(X)} \circ F(\delta_X) = Id_{F(X)} \text{ and } \delta_{G(E)} \circ G(\gamma_E) = Id_{G(E)}. \square$$

1.2.5 Unicity

Proposition 1.14 *Let $\mathcal{C}$ be a Galois category and $F,\ F' : \mathcal{C} \to FSets$ two fibre functors defining profinite groups $\Pi := \pi_1(\mathcal{C}; F)$ and $\Pi' := \pi_1(\mathcal{C}; F')$ associated with universal coverings $(\phi_{g,h} : (X_g, \zeta_g) \to (X_h, \zeta_h))_{g,h \in \mathcal{G},\, g \geq h}$ and $(\phi'_{g',h'} : (X'_{g'}, \zeta'_{g'}) \to (X'_{h'}, \zeta'_{h'}))_{g,h \in \mathcal{G}',\, g' \geq h'}$ respectively. Then there is a profinite group isomorphism $\Pi \tilde{\to} \Pi'$ canonical up to inner automorphisms.*

Proof. From proposition 1.13, one may assume that $\mathcal{C}$ is $\mathcal{C}(\Pi)$, $F : \mathcal{C} \to FSets$ is the forgetful functor $For : \mathcal{C}(\Pi) \to FSets$ and $(\phi_{g,h} : (X_g, \zeta_g) \to (X_h, \zeta_h))_{g,h \in \mathcal{G},\, g \geq h}$ is the projective system induced by the normal open subgroups of Π pointed by the identity element 1 *i.e.* $(\phi_{N,M} : (\Pi/N, 1) \to (\Pi/M, 1))_{N<M,\ N,\ M \text{ normal open subgroups in } \Pi}$. For each $g' \in \mathcal{G}'$ let $\alpha_{g'} : \Pi \to F'(X'_{g'})$ be the morphism of $Pro(\mathcal{C}(\Pi))$ defined by $\alpha_{g'}(1) = \zeta'_{g'}$. Since $X'_{g'}$ is connected, $\alpha_{g'} : \Pi \to F'(X'_{g'})$ is an epimorphism in $Pro(\mathcal{C}(\Pi))$ hence so is $\underline{\alpha} := \varprojlim \alpha_{g'} : \Pi \twoheadrightarrow \varprojlim F'(X'_{g'})$. Write $\underline{\zeta}' := (\zeta'_{g'})_{g' \in \mathcal{G}'} \in \varprojlim F'(X'_{g'})$. Then

$\mathrm{Stab}_\Pi(\underline{\zeta}') \subset \Pi$ is a closed subgroup such that $\underline{\alpha} : \Pi \to \varprojlim F'(X'_{g'})$ factors through an isomorphism in $Pro(\mathcal{C}(\Pi))$:

$$\begin{tikzcd} \Pi \arrow[r, two heads, "\underline{\alpha}"] \arrow[d] & \varprojlim F'(X'_{g'}). \\ \Pi/\mathrm{Stab}_\Pi(\underline{\zeta}') \arrow[ur, hook, two heads, "\overline{\underline{\alpha}}"'] & \end{tikzcd}$$

The functor isomorphism $\mathrm{Hom}_{Pro(\mathcal{C}(\Pi))}(\varprojlim F'(X'_{g'}), -)\tilde{\to}\mathrm{Hom}_{Pro(\mathcal{C}(\Pi))}(\Pi/\mathrm{Stab}_\Pi(\underline{\zeta}'), -)$, $u \mapsto u \circ \overline{\underline{\alpha}}$ thus identifies F' with the functor $X \mapsto X^{\mathrm{Stab}_\Pi(\underline{\zeta}')}$. In particular, since $F'(X) = \emptyset$ if and only if $X = \emptyset$ and since for any normal open subgroup $N \lhd \Pi$, $\Pi/N \neq \emptyset$, one gets $F'(\Pi/N) = \Pi/N^{\mathrm{Stab}_\Pi(\underline{\zeta}')} \neq \emptyset$. Hence $\varprojlim \Pi/N^{\mathrm{Stab}_\Pi(\underline{\zeta}')} = \Pi^{\mathrm{Stab}_\Pi(\underline{\zeta}')} \neq \emptyset$, which forces $\mathrm{Stab}_\Pi(\underline{\zeta}') = 1$. As a result, $\underline{\alpha} : \Pi\tilde{\to}\varprojlim F'(X'_{g'})$ is an isomorphism in $Pro(\mathcal{C}(\Pi))$ and one gets a profinite group isomorphism:

$$\begin{array}{rccc} \Phi_{\underline{\alpha}} : & \Pi'^{op} = \mathrm{Aut}_{Pro(\mathcal{C}(\Pi))}(\underline{X}') & \tilde{\to} & \mathrm{Aut}_{Pro(\mathcal{C}(\Pi))}(\Pi) = \Pi^{op} \\ & \sigma & \mapsto & \underline{\alpha}^{-1}\sigma\underline{\alpha}. \end{array}$$

Furthermore $\underline{\alpha}$ (hence $\Phi_{\underline{\alpha}}$ is uniquely determined by $\underline{\zeta}'$ and replacing $\underline{\zeta}'$ by $\underline{\zeta}''$ amounts to replacing $\underline{\alpha}$ by $\underline{\alpha}'$ and $\Phi_{\underline{\alpha}}$ by $(\underline{\alpha}^{-1}\underline{\alpha}')^{-1}\Phi_{\underline{\alpha}}(-)(\underline{\alpha}^{-1}\underline{\alpha}')$ (with $\underline{\alpha}^{-1}\underline{\alpha}' \in \Pi^{op}$). $\Box$

1.3 Fundamental functors and functoriality

Let $\mathcal{C}$ be a Galois category. Then, given a fibre functor $F : \mathcal{C} \to FSets$, we fix a universal covering $(\phi_{F,g,h} : (X_{F,g}, \zeta_{F,g}) \to (X_{F,h}, \zeta_{F,h}))_{g,\, h \in \mathcal{G}_F,\, g \geq h}$ for $F : \mathcal{C} \to FSets$.

1.3.1 Fundamental functors

Proposition 1.15 *Given Galois categories $\mathcal{C}$, $\mathcal{C}'$ and a covariant functor $H : \mathcal{C} \to \mathcal{C}'$, the following assertions are equivalent.*
(i) There exists a fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$ such that $F' \circ H : \mathcal{C} \to FSets$ is a fibre functor for $\mathcal{C}$.
(ii) For all fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$, $F' \circ H : \mathcal{C} \to FSets$ is a fibre functor for $\mathcal{C}$.
(iii) $H : \mathcal{C} \to \mathcal{C}'$ is exact (that is is left exact and right exact).

Proof. Let us show that $(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$. The implication $(ii) \Rightarrow (i)$ is straightforward.

$(i) \Rightarrow (iii)$. One has to prove that H commutes with kernels and cokernels. So, let $u_1,\ u_2 : X \rightrightarrows Y$ be morphisms in $\mathcal{C}$ and let $F' : \mathcal{C}' \to FSets$ be a fibre functor for $\mathcal{C}'$ such that $F' \circ H : \mathcal{C} \to FSets$ is a fibre functor for $\mathcal{C}$. Since $F' \circ H$ commutes with kernels, the sequence

$$F' \circ H(\ker(u_1, u_2)) \overset{F'\circ H(i)}{\to} F' \circ H(X) \overset{F'\circ H(u_1), F'\circ H(u_2)}{\rightrightarrows} F' \circ H(Y) \quad (*)$$

is exact is $FSets$. On the other hand, $H(u_1)\circ H(i) = H(u_2)\circ H(i)$, whence a canonical factorization in $\mathcal{C}'$:

$$\begin{tikzcd} \ker(H(u_1),H(u_2)) \arrow[r] & H(X) \arrow[r, "{H(u_1),H(u_2)}"] & H(Y) \quad (**) \\ & H(\ker(u_1,u_2)) \arrow[ul, dotted, "\nu"] \arrow[u, "H(i)"'] & \end{tikzcd}$$

But, in view of $(*)$ applying F' to $(**)$, one gets that $F'(\nu)$ is an isomorphism in Fsets hence, by axiom (6), $\nu : H(\ker(u_1,u_2))\tilde{\to}\ker(H(u_1),H(u_2))$ is an isomorphism in $\mathcal{C}'$. The same argument shows that $H:\mathcal{C}\to\mathcal{C}'$ commutes with cokernels.

$(iii)\Rightarrow(ii)$. Assume now that $H:\mathcal{C}\to\mathcal{C}'$ is exact and let $F':\mathcal{C}'\to FSets$ be a fibre functor for $\mathcal{C}'$. Then $F'\circ H:\mathcal{C}\to FSets$ is again exact. So it only remains to check axiom (6) for $F'\circ H:\mathcal{C}\to FSets$. This will follow from the fact that, if $X\neq\emptyset_{\mathcal{C}}$ then $H(X)\neq\emptyset_{\mathcal{C}'}$. Indeed, in general, $X\neq\emptyset_{\mathcal{C}}$ if and only if the canonical morphism $v_X : X\to e_{\mathcal{C}}$ is an epimorphism. But as $H:\mathcal{C}\to\mathcal{C}'$ is right exact, it transforms epimorphisms into epimorphisms and as it is left exact it transforms $e_{\mathcal{C}}$ into $e_{\mathcal{C}'}$. Now, let $u:X\to Y$ be a morphism in $\mathcal{C}$ such that $F'\circ H(u) : F'\circ H(X)\tilde{\to}F'\circ H(Y)$ is an isomorphism in FSets. Hence, by axioms (6) applied to $F':\mathcal{C}'\to FSets$, $H(u):H(X)\tilde{\to}H(Y)$ is an isomorphism in $\mathcal{C}$. From axiom (3), u factors as $u : X\overset{u'}{\to}Y'\overset{u''}{\to}Y = Y'\coprod Y''$ with u' a strict epimorphism and u'' a monomorphism inducing an isomorphism onto the direct factor Y' of Y. Since $H:\mathcal{C}\to\mathcal{C}'$ is exact, the factorization $H(u):H(X)\overset{H(u')}{\to}H(Y')\overset{H(u'')}{\to}H(Y)=H(Y')\coprod H(Y'')$ is again the one given by axiom (3) for $H(u)$ in $\mathcal{C}'$. In particular $H(Y'')=\emptyset_{\mathcal{C}'}$ hence $Y''=\emptyset_{\mathcal{C}}$ and $u:X\to Y$ is a strict epimorphism. Assume it is not a monomorphism. Then there exists two distinct morphisms in $\mathcal{C}$ $u_i : W\to X$, $i=1,2$ such that $u_1\circ u = u_2\circ u$. Since $H(u_1)\circ H(u) = H(u_1\circ u) = H(u_2\circ u) = H(u_2)\circ H(u)$ and $H(u) : H(X)\tilde{\to}H(Y)$ is an isomorphism, $H(u_1)=H(u_2)$ hence $\ker(H(u_1),H(u_2)) = H(X)$. But, as well, $\ker(H(u_1),H(u_2)) = H(\ker(u_1,u_2))$. So, if $i:\ker(u_1,u_2)\hookrightarrow X$ denotes the canonical monomorphism, $H(i):H(\ker(u_1,u_2))\tilde{\to}H(X)$ is an isomorphism hence, by the argument above, $i:\ker(u_1,u_2)\hookrightarrow X$ is also a strict epimorphism thus, by lemma 1.4 (ii), an isomorphism, which contradicts the fact that u_1 and u_2 are distinct. So $u:X\to Y$ is also a monomorphism hence an isomorphism. $\square$

A functor $H:\mathcal{C}\to\mathcal{C}'$ satisfying properties (i), (ii), (iii) of proposition 1.15 is called a *Fundamental functor from $\mathcal{C}$ to $\mathcal{C}'$.*

Let $u:\Pi'\to\Pi$ be a profinite groups morphism. Then any $E\in\mathcal{C}(\Pi)$ can be endowed with a continuous action of Π' *via* $u:\Pi'\to\Pi$, which defines a canonical fundamental functor:

$$H_u : \mathcal{C}(\Pi)\to\mathcal{C}(\Pi').$$

Conversely, let $H:\mathcal{C}\to\mathcal{C}'$ be a fundamental functor. Let $F':\mathcal{C}'\to FSets$ be a fibre functor for $\mathcal{C}'$, $F:=F'\circ H:\mathcal{C}\to FSets$ and set $\Pi:=\pi_1(\mathcal{C};F)$, $\Pi':=\pi_1(\mathcal{C}';F')$. Then for any $\Theta'\in\Pi'$, $\Theta'\circ H\in\Pi$, which defines a canonical group morphism:

$$u_H:\Pi'\to\Pi,$$

which is continuous since for any $X\in\mathcal{C}$, $u_H^{-1}(K_X) = K_{H(X)}$. And one immediately checks that:

$$u_{H_u} = u : \Pi'\rightrightarrows\Pi$$

and that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(\Pi) \arrow[r, "H_{u_H}"] & \mathcal{C}(\Pi') \\
\mathcal{C} \arrow[u, "F"] \arrow[r, "H"'] & \mathcal{C}'. \arrow[u, "F'"']
\end{tikzcd}$$

Thus a functor $H : \mathcal{C} \to \mathcal{C}'$ is a fundamental functor if and only if there exists a profinite groups morphism $u : \Pi' \to \Pi$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(\Pi) \arrow[r, "H_{u}"] & \mathcal{C}(\Pi') \\
\mathcal{C} \arrow[u, "F"] \arrow[r, "H"'] & \mathcal{C}'. \arrow[u, "F'"']
\end{tikzcd}$$

In the next §, we are going to compare the properties of the fundamental functor $H : \mathcal{C} \to \mathcal{C}'$ and of the corresponding profinite group morphism $u : \Pi' \to \Pi$.

Example 1.16

1. Any field extension $\phi : k \to k'$ defines a canonical functor

$$\begin{array}{rlcl} H: & FSA_k & \to & FSA_{k'} \\ & k \hookrightarrow A & \mapsto & k' \hookrightarrow A \otimes_{k,\phi} k'. \end{array}$$

and for any algebraically closed field extension $i' : k' \hookrightarrow \Omega$, one has:

$$\begin{array}{rl} F_{i'} \circ H & = \mathrm{Hom}_{FSA_{k'}}(- \otimes_{k,\phi} k', \Omega) \\ & \overset{(*)}{=} \mathrm{Hom}_{FSA_k}(-, \Omega) = F_{i' \circ \phi}, \end{array}$$

where the equality (*) comes from the universal property of tensor product. Hence $H : FSA_k \to FSA_{k'}$ is a fundamental functor. In that case, the corresponding profinite groups morphism is just the restriction morphism:

$$|_{k^s} : \Gamma_{k'} \to \Gamma_k.$$

2. Any continuous map $\phi : B' \to B$ of connected, locally arcwise connected and locally simply connected topological spaces defines a canonical functor:

$$\begin{array}{rlcl} H: & FR_B^{top} & \to & FR_{B'}^{top} \\ & f : X \to B & \mapsto & p_2 : X \times_{f,B,\phi} B' \to B'. \end{array}$$

and for any $b' \in B'$, one has:

$$\begin{array}{rl} F_{b'} \circ H(f) & = p_2^{-1}(b') \\ & = \{(x, b') \mid x \in X \text{ such that } f(x) = \phi(b')\} \\ & = f^{-1}(\phi(b')). \end{array}$$

Hence $H : FR_B \to FR_{B'}$ is a fundamental functor. In that case, the corresponding profinite groups morphism is just the canonical morphism:

$$\hat{\phi} : \pi_1^{top}\widehat{(B', b')} \to \pi_1^{top}\widehat{(B, \phi(b'))}$$

induced from $\phi : \pi_1^{top}(B', b') \to \pi_1^{top}(B, \phi(b'))$.

1.3.2 Functoriality

Lemma 1.17 *With the above notation:*

1. *For any open subgroup* $S \subset \Pi$, $\mathrm{Im}(u)) \subset S$ *(resp.* $\mathrm{Nor}_\Pi(\mathrm{Im}(u)) \subset S$*) if and only if* $\mathrm{Hom}_{\mathcal{C}^{pt}}((e_\mathcal{C}, *), (H(\Pi/S), 1)) \neq \emptyset$ *(resp.* $H(\Pi/S)$ *is totally split in* $\mathcal{C}$*). In particular,* $u : \Pi' \to \Pi$ *is trivial if and only if for any object* X *in* $\mathcal{C}$, $H(X)$ *is totally split in* $\mathcal{C}'$*.*

2. *For any open subgroup* $S' \subset \Pi'$, $\mathrm{Ker}(u) \subset S'$ *if and only if there exists an open subgroup* $S \subset \Pi$ *such that* $\mathrm{Hom}_{\mathcal{C}^{pt}}((H(\Pi/S), 1)_0, (\Pi'/S', 1)) \neq \emptyset$*(where, for any* $(X, \zeta) \in \mathcal{C}^{pt}$, $(X, \zeta)_0 = (X_0, \zeta)$*, where* X_0 *denotes the connected component of* ζ *in* X*). If* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $\mathrm{Ker}(u) \subset S'$ *if and only if there exists an open subgroup* $S \subset \Pi$ *such that* $\mathrm{Isom}_{\mathcal{C}^{pt}}((H(\Pi/S), 1)_0, (\Pi'/S', 1)) \neq \emptyset$*. In particular,* $u : \Pi' \hookrightarrow \Pi$ *is a monomorphism if and only if for any connected object* $X' \in \mathcal{C}'$ *there exists a connected object* $X \in \mathcal{C}$ *and a connected component* $H(X)_0$ *of* $H(X)$ *in* $\mathcal{C}$ *such that* $\mathrm{Isom}_{\mathcal{C}'}((H(X)_0, X') \neq \emptyset$*. If, furthermore,* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $u : \Pi' \twoheadrightarrow \Pi$ *is an isomorphism if and only if* $H : \mathcal{C} \to \mathcal{C}'$ *is essentially surjective.*

Proof. Recall that, given a profinite group Π, a closed subgroup $S \subset \Pi$ is the intersection of all open subgroup $U \subset \Pi$ containing S thus, in particular, $\{e\}$ is the intersection of all open subgroups of Π. This yields the characterization of trivial morphisms and monomorphisms from the preceding assertions in (1) and (2).

For the first assertion of (1), if $\mathrm{Im}(u) \subset S$ then for any $\Theta' \in \Pi'$ $\Theta' \cdot S = u(\Theta')S = S$ hence the canonical inlcusion $\Pi'/\Pi' \hookrightarrow H(\Pi/S) = \Pi'/u^{-1}(S)$ in $\mathcal{C}(\Pi')$ induces a morphim $(\Pi'/\Pi', 1) \hookrightarrow (H(\Pi/S), 1)$ in $\mathcal{C}'^{pt}$. Conversely, if $\mathrm{Hom}_{\mathcal{C}^{pt}}((e_\mathcal{C}, *), (H(\Pi/S), 1)) \neq \emptyset$ then let $u : (\Pi'/\Pi', 1) \to (H(\Pi/S), 1)$ in $\mathcal{C}'^{pt}$. For any $\Theta' \in \Pi'$, one has $\Theta' \cdot 1 = \Theta' \cdot u(1) = u(\Theta') = 1$ so $\mathrm{Im}(u)) \subset S$.

For (2), if $\mathrm{Ker}(u) \subset S'$ then one has a canonical isomorphism $(\Pi'/S', 1)\tilde{\to}(\mathrm{Im}(u)/u(S'), 1)$ in $\mathcal{C}'^{pt}$. In particular, since both Π' and S' are compact, $u(S') \subset \mathrm{Im}(u)$ is a closed subgroup of finite index in $\mathrm{Im}(u)$ hence is also open in $\mathrm{Im}(u)$ and there exists an open subgroup $S \subset \Pi$ such that $S \cap \mathrm{Im}(u) \subset u(S')$. By definition, the connected component of 1 in $H(\Pi/S)$ in $\mathcal{C}'$ is $\mathrm{Im}(u)S/S \simeq \mathrm{Im}(u)/S \cap \mathrm{Im}(u) \simeq \Pi'/u^{-1}(S)$. But $u^{-1}(S) = u^{-1}(S \cap \mathrm{Im}(u)) \subset S'$, whence a canonical morphism $(\mathrm{Im}(u)S/S, 1) \to (\Pi'/S', 1)$ in $\mathcal{C}'^{pt}$. Conversely, assume that there exists an open subgroup $S \subset \Pi$ and a morphism $(\mathrm{Im}(u)S/S, 1) \to (\Pi'/S', 1)$ in $\mathcal{C}'^{pt}$ then, by definition of pointed morphisms, $\mathrm{Ker}(u) \subset u^{-1}(S) \subset S'$. If $\mathrm{Im}(u) = \Pi$, one can take $S = u(S')$. Eventually, note that since $\mathrm{Ker}(u) \lhd \Pi'$ is normal in Π', the condition $\mathrm{Ker}(u) \subset S'$ does not depend on the choice of $\zeta \in F(X)$ defining the isomorphism $X'\tilde{\to}\Pi'/S'$. □

Proposition 1.18

1. *The following three assertions are equivalent:*
   (i) $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism;*
   (ii) $H : \mathcal{C} \to \mathcal{C}'$ *sends connected objects to connected objects;*
   (iii) $H : \mathcal{C} \to \mathcal{C}'$ *is fully faithful.*

2. $u : \Pi' \hookrightarrow \Pi$ *is a monomorphism if and only if for any object* X' *in* $\mathcal{C}'$ *there exists an object* X *in* $\mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *such that* $\mathrm{Hom}_{\mathcal{C}'}(X'_0, X') \neq \emptyset$*.*

3. $u : \Pi' \tilde{\to} \Pi$ *is an isomorphism if and only if* $H : \mathcal{C} \approx \mathcal{C}'$ *is an equivalence of categories.*

4. *If* $\mathcal{C} \xrightarrow{H} \mathcal{C}' \xrightarrow{H'} \mathcal{C}''$ *is a sequence of fundamental functors of Galois categories with corresponding sequence of profinite groups* $\Pi \xleftarrow{u} \Pi' \xleftarrow{u'} \Pi''$*. Then,*
- $\mathrm{Ker}(u) \subset \mathrm{Im}(u')$ *if and only if for any object* X *in* $\mathcal{C}$ $H'(H(X))$ *is totally split in* $\mathcal{C}''$*;*
- $\mathrm{Ker}(u) \supset \mathrm{Im}(u')$ *if and only if for any connected object* X' *in* $\mathcal{C}'$ *such that* $H'(X')$ *admits a section, there exists an object* X *in* $\mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *such that* $\mathrm{Hom}_{\mathcal{C}'}(X'_0, X') \neq \emptyset$*.*

Proof. Assertion (2) follows from lemma 1.17. Assertions (3) and (4) follow from lemma 1.17 and (1). So we are only to prove (1).
(i) $\Rightarrow$ (ii). Assume that $u : \Pi' \twoheadrightarrow \Pi$ is an epimorphism. Then, for any connected object X in $\mathcal{C}(\Pi)$, Π acts transitively on X. But $H(X)$ is just X equipped with the Π'-action $\sigma' \cdot x = u(\sigma') \cdot x$. Hence Π' acts transitively on $H_u(X)$ as well or, equivalently, $H(X)$ is connected.
(ii) $\Rightarrow$ (i). Assume that for any connected object X in $\mathcal{C}(\Pi)$ $H_u(X)$ is again connected in $\mathcal{C}(\Pi')$. This holds, in particular, for any finite quotient Π/N of Π (with $N \lhd \Pi$ a normal open subgroup) that is, the canonical morphism $u_N : \Pi' \xrightarrow{u} \Pi \overset{pr_N}{\twoheadrightarrow} \Pi/N$ is a continuous epimorphism hence so is $u = \varprojlim u_N$.
(i) $\Leftrightarrow$ (iii) is straightforward. $\square$

Given a Galois category $\mathcal{C}$ with fibre functor $F : \mathcal{C} \to FSets$ and $X \in \mathcal{C}$ connected, let $\mathcal{C}_X$ denote the *category of X-objects* that is the category defined by:
- Objects: Morphism $f : Y \to X$ in $\mathcal{C}$;
- Morphisms from $f' : Y' \to X$ to $f : Y \to X$:

$$\mathrm{Hom}_{\mathcal{C}_X}(f', f) = \{\phi : Y' \to Y \in \mathrm{Hom}_{\mathcal{C}}(Y', Y) \mid f \circ \phi = f'\}.$$

And for any $\zeta \in F(X)$, set

$$\begin{array}{llll} F_{(X,\zeta)} : & \mathcal{C}_X & \to & FSets \\ & f : Y \to X & \mapsto & F(f)^{-1}(\zeta). \end{array}$$

Proposition 1.19 *$\mathcal{C}_X$ is Galois with fibre functors $F_{(X,\zeta)} : \mathcal{C}_X \to FSets$, $\zeta \in F(X)$. Furthermore, the canonical functor*

$$\begin{array}{llll} H : & \mathcal{C} & \to & \mathcal{C}_X \\ & Y & \mapsto & p_2 : Y \times X \to X \end{array}$$

satisfies, for any $\zeta \in F(X)$ $F_{(X,\zeta)} \circ H = F$ *and induces a profinite group monomorphism:* $\pi_1(\mathcal{C}_X; F_{(X,\zeta)}) \hookrightarrow \pi_1(\mathcal{C}; F)$ *with image* $\mathrm{Stab}_{\pi_1(\mathcal{C};F)}(\zeta)$*.*

Proof. Just observe that if $(\phi_{F,g,h} : (X_{F,g}, \zeta_{F,i}) \to (X_{F,h}, \zeta_{F,h}))_{g,\, h \in \mathcal{G}_F,\, g \geq h}$ is a universal covering for F then with $\mathcal{G}_{F_{(X,\zeta)}} \subset \mathcal{G}_F$ the set of all $g \in \mathcal{G}_F$ such that $\mathrm{Hom}_{\mathcal{C}^{pt}}((X_g, \zeta_g), (X, \zeta)) \neq \emptyset$ defines a universal covering $(\phi_{F,g,h} : (X_{F,g}, \zeta_{F,g}) \to (X_{F,h}, \zeta_{F,h}))_{g,\, h \in \mathcal{G}_{F_{(X,\zeta)}},\, g \geq h}$ for $F_{(X,\zeta)}$. Furthermore, for any $g \in \mathcal{G}_{F_{(X,\zeta)}}$, identify as usual $(X_{F,g}, \zeta_{F,g}) \to (X, \zeta)$ with the canonical morphism $\pi_1(\mathcal{C}, F)/\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta_{F,g}) \to \pi_1(\mathcal{C}, F)/\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta)$ and, thus, $F_{(X,\zeta)}(X_{F,g})$ with $\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta)/\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta_{F,g})$. Through the above identification, $\pi_1(\mathcal{C}_X, F_{(X,\zeta)}) = \varprojlim F_{(X,\zeta)}(X_{F,g}) = \varprojlim \mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta)/\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta_{F,g}) = \mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta)$. Also, $F_{(X,\zeta)} \circ H \simeq F$ thus $H : \mathcal{C} \to \mathcal{C}_X$ is exact and $u_H : \pi_1(\mathcal{C}_X, F_{(X,\zeta)}) \to \pi_1(\mathcal{C}, F)$ is nothing but the canonical inclusion $\mathrm{Stab}_{\pi_1(\mathcal{C},F)}(\zeta) \hookrightarrow \pi_1(\mathcal{C}, F)$. $\square$

2 Examples

2.1 Topological covers

The formalism of Galois categories is derived from what occurs for topological covers. In the following, all the topological spaces will be assumed to be separated.

Recall that the *topological fundamental group of X based at $x \in X$* is the group $\pi_1^{top}(X,x)$ of homotopy (with x fixed) classes of closed paths based at x. If X is arcwise connected then, given two points $x_0,\ x_1 \in X$, any path $c:[0,1]\to X$ from x_0 to x_1 defines a group isomorphism $\pi_1^{top}(X,x_0)\tilde{\to}\pi_1^{top}(X,x_1), [\gamma]\mapsto [c][\gamma][c]^{-1}$.

Example 2.1 Topological groups can sometimes be computed explicitly.

1. If X is a compact Riemann surface with genus g menus r points then $\pi_1^{top}(X,x)=\Gamma_{g,r}$, where $\Gamma_{g,r}$ is the group with generators $\gamma_1,\dots,\gamma_r,\ u_1,\dots,u_g,v_1,\dots,v_g$ and single relation
$$\gamma_1\cdots\gamma_r[u_1,v_1]\cdots[u_g,v_g]=1.$$
Note in particular that the only cases when $\pi_1^{top}(X,x)$ is abelian are for $(g,r)=(0,0),(0,1),(0,2),(1,0)$. These are special cases of example (3) below.
2. If $X=U_r$ is the configuration space for r unordered points on the projective line then $\pi_1^{top}(X,x)=H_r$ is the so-called Hurwitz braid group given by the generators $Q_1,...,Q_{r-1}$ and defining relations
$$\begin{array}{lll} (1) & Q_iQ_{i+1}Q_i=Q_{i+1}Q_iQ_{i+1} & \text{for } i=1,...,r-2 \\ (2) & Q_iQ_j=Q_jQ_i & \text{for } i,j=1,...,r-1 \text{ with } |j-i|>1 \\ (3) & Q_1Q_2\cdots Q_{r-1}Q_{r-1}\cdots Q_2Q_1=1 & \end{array}$$
3. If X is a topological group then $\pi_1^{top}(X,x)$ is abelian.

An arcwise connected topological space with trivial topological fundamental group is said to be *simply connected*.

Given a topological space X, let R_X^{top} denote the category of topological covers of X and, for any $x\in X$, let $R_{(X,x)}^{top}$ denote the associated pointed category. Also, write $F_x: R_X^{top}\to Sets$ for the functor sending $p:Y\to X\in R_X^{top}$ to $F_x(p)=p^{-1}(x)$. Then F_x naturally factors through the category $\mathcal{C}^{disc}(\pi_1^{top}(X,x))$ of (discrete) $\pi_1^{top}(X,x)$-sets. The natural action of $\pi_1^{top}(X,x)$ on $F_x(p)$ is given by monodromy.

Lemma 2.2 (monodromy) *For any $p:Y\to X\in R_X^{top}$, any path $c:[0,1]\to X$ and any $y\in F_{c(0)}(p)$, there exists a unique path $\tilde{c}_y:[0,1]\to Y$ such that $p\circ\tilde{c}_y=c$ and $\tilde{c}_y(0)=y$. Furthermore, if $c_1,c_2:[0,1]\to X$ are two homotopic paths with fixed ends then $\tilde{c}_{1,y}(1)=\tilde{c}_{2,y}(1)$.*

In particular, one gets a well defined action $\rho_x(p):\pi_1^{top}(X,x)\to \mathrm{Aut}_{Sets}(F_x(p))$ sending $[\gamma]\in\pi_1^{top}(X,x)$ to $\rho_x(p)([\gamma]): y\mapsto\tilde{\gamma}_y(1)$ and ρ_x defines a group morphism $\rho_x:\pi_1^{top}(X,x)\to\mathrm{Aut}_{Fct}(F_x)$, $[\gamma]\mapsto\rho_x(-)([\gamma])$.

And, actually, one has:

Proposition 2.3 *Assume that X is connected, locally arcwise connected and locally simply connected. Then $F_x: R_X^{top}\to Sets$ induces an equivalence of categories*
$$F_x: R_X^{top}\approx\mathcal{C}^{disc}(\pi_1^{top}(X,x)).$$

Proof (sketch of). A *universal covering for X pointed at x* is an element $(\tilde{p}_X : \tilde{X} \to X, \tilde{x}) \in R^{top}_{(X,x)}$ such that for any $(p : Y \to X, y) \in R^{top}_{(X,x)}$, there exists a unique morphism from $(\tilde{p}_X, \tilde{x})$ to (p, y) in $R^{top}_{(X,x)}$.

Step 0: basic facts about topological coverings.

1. For any $(p_i : Y_i \to X, y_i) \in R^{top}_{(X,x)}$ with Y_i connected, $i = 1,\ 2$ $\mathrm{Hom}_{R^{top}_{(X,x)}}((p_1, y_1), (p_2, y_2)) \neq \emptyset$ if and only if $p_1 \circ (\pi_1(Y_1, y_1)) \subset p_2 \circ (\pi_1(Y_2, y_2))$ and $\mathrm{Isom}_{R^{top}_{(X,x)}}((p_1, y_1), (p_2, y_2)) \neq \emptyset$ if and only if $p_1 \circ (\pi_1(Y_1, y_1)) = p_2 \circ (\pi_1(Y_2, y_2))$.
   In particular, any $(p : Y \to X, y) \in R^{top}_{(X,x)}$ with Y simply connected is a universal covering for X pointed at x.
2. For any $(p : Y \to X, y) \in R^{top}_{(X,x)}$ with Y connected, $p : Y \to X$ is Galois if and only if $p \circ (\pi_1(Y, y))$ is normal in $\pi_1^{top}(X, x)$.
   In particular, if Y is simply connected then $p : Y \to X$ is automatically Galois.
3. For any $(p : Y \to X, y) \in R^{top}_{(X,x)}$ with Y connected, one can show that for any $[\gamma] \in \mathrm{Nor}_{\pi_1^{top}(X,x)}(p \circ (\pi_1^{top}(Y, y)))$ there exists a unique $u([\gamma]) \in \mathrm{Aut}_{R^{top}_X}(p)$ such that $p(y) = \tilde{\gamma}_y(1)$. This defines a group morphism $u : \mathrm{Nor}_{\pi_1^{top}(X,x)}(p \circ (\pi_1^{top}(Y, y))) \to \mathrm{Aut}_{R^{top}_X}(p)$ which fits in the following canonical short exact sequence:
   $$1 \to \pi_1^{top}(Y, y) \xrightarrow{p\circ} \mathrm{Nor}_{\pi_1^{top}(X,x)}(p \circ (\pi_1(Y, y))) \xrightarrow{u} \mathrm{Aut}_{R^{top}_X}(p) \to 1.$$
   In particular, if Y is simply connected then $u : \pi_1^{top}(X, x) \tilde{\to} \mathrm{Aut}_{R^{top}_X}(p)$ is an isomorphism.

Step 1: Universal covering. The hypotheses on X ensure the existence of universal coverings. They can be explicitly constructed as follows. Let $\tilde{X}$ denote the set of all paths $c : [0, 1] \to X$ with $c(0) = x$ modulo homotopy with fixed ends. One thus gets a well-defined map $\tilde{p}_X : \tilde{X} \to X$ sending $[c] \in \tilde{X}$ to $p([c]) = c(1)$. $\tilde{X}$ can be endowed with a topology in such a way that $\tilde{p}_X : \tilde{X} \to X$ becomes a topological cover. For any $[c] \in \tilde{X}$, let $U_{\tilde{p}([c])}$ be a simply connected open neighborhood of $\tilde{p}([c]) \in X$. Since $U_{\tilde{p}([c])}$ is simply connected, for any $u \in U_{\tilde{p}([c])}$ there exists a path $(\tilde{p}([c]), u) : [0, 1] \to X$, unique up to homotopy with fixed ends, from $\tilde{p}([c])$ to u. This yields a well-defined map $\Phi_{U_{\tilde{p}([c])}} : \tilde{p}^{-1}(U_{\tilde{p}([c])}) \to U_{\tilde{p}([c])} \times \tilde{p}^{-1}(\tilde{p}([c]))$ sending $[c']$ to $(\tilde{p}([c']), [(\tilde{p}([c]), \tilde{p}([c']))][c'])$, which is actually bijective (with inverse map the map $U_{\tilde{p}([c])} \times \tilde{p}^{-1}(\tilde{p}([c])) \to \tilde{p}^{-1}(U_{\tilde{p}([c])})$ sending $(u, [c'])$ to $[c'][(u, \tilde{p}([c']))]$). Furthermore, for two simply connected open neighborhoods $U^1_{\tilde{p}([c])}$, $U^2_{\tilde{p}([c])}$ of $\tilde{p}([c])$, $\Phi_{U^1_{\tilde{p}([c])}} \circ \Phi^{-1}_{U^2_{\tilde{p}([c])}} : U^1_{\tilde{p}([c])} \cap U^2_{\tilde{p}([c])} \times \tilde{p}^{-1}(\tilde{p}([c]) \to U^1_{\tilde{p}([c])} \cap U^2_{\tilde{p}([c])} \times \tilde{p}^{-1}(\tilde{p}([c])$ is continous. Hence there exists a unique topology on $\tilde{X}$ such that the $\tilde{p}^{-1}(U_{\tilde{p}([c])})$ are open and the $\Phi_{U_{\tilde{p}([c])}} : \tilde{p}^{-1}(U_{\tilde{p}([c])}) \to U_{\tilde{p}([c])} \times \tilde{p}^{-1}(\tilde{p}([c]))$ are homeomorphisms. For this topology $\tilde{X}$ is separated, simply connected and $\tilde{p} : \tilde{X} \to X$ is a topological cover. Furthermore the isomorphism $u : \pi_1(X, x) \tilde{\to} \mathrm{Aut}_{R^{top}_X}(\tilde{p})$ is just given by the translation $\pi_1(X, x) \times \tilde{X} \to \tilde{X}$, $([\gamma], [c]) \mapsto [c][\gamma]$.

Step 2: F_x is essentially surjective. Since F_x commutes with finite coproduct, it is enough to show that the connected objects in $\mathcal{C}^{disc}(\pi_1^{top}(X, x))$ are in the essential image of F_x. So let E be a transitive $\pi_1^{top}(X, x)$-set. Then, for any $e \in E$ one has $\mathrm{Stab}_{\pi_1^{top}(X,x)}(e) \subset \pi_1^{top}(X, x) = \mathrm{Aut}_{R^{top}_{(X,x)}}(\tilde{p})$ and writing the corresponding quotient cover $\tilde{p}_{Stab_{\pi_1^{top}(X,x)}(e)} : \tilde{X}/\mathrm{Stab}_{\pi_1^{top}(X,x)}(e) \to X$ one check that $F_x(\tilde{p}_{Stab_{\pi_1^{top}(X,x)}(e)})$ and E are isomorphic in $\mathcal{C}^{disc}(\pi_1^{top}(X, x))$.

Step 3: F_x is fully faithfull. From step 2, this amount to showing that for any subgroups $N_1,\ N_2 \subset \pi_1^{top}(X,x)$, the canonical map

$$F_x : \mathrm{Hom}_{R_X^{top}}(\tilde{p}_{N_1}, \tilde{p}_{N_2}) \to \mathrm{hom}_{\mathcal{C}^{disc}(\pi_1^{top}(X,x))}(\pi_1^{top}(X,x)/N_1, \pi_1^{top}(X,x)/N_2)$$

is bijective. For this, just observe that the map $\Phi_x : \mathrm{Hom}_{\mathcal{C}^{disc}(\pi_1^{top}(X,x))}(\pi_1^{top}(X,x)/N_1, \pi_1^{top}(X,x)/N_2) \to \mathrm{hom}_{R_X^{top}}(\tilde{p}_{N_1}, \tilde{p}_{N_2})$ sending $f : \pi_1^{top}(X,x)/N_1 \to \pi_1^{top}(X,x)/N_2$ to $\Phi_x(f) : \tilde{X}/N_1 \to \tilde{X}/N_2$, $[c]N_1 \mapsto [c]f(1)N_2$ is an inverse for F_x. $\square$

Corollary 2.4 *Assume that X is connected, locally arcwise connected and locally simply connected. Then the group morphism $\rho_x : \pi_1(X,x) \to \mathrm{Aut}_{Fct}(F_x)$ is an isomorphism.*

Also, the following corollary immediately follows from proposition 2.3.

Corollary 2.5 *Assume that X is connected, locally arcwise connected and locally simply connected. Then the category R_X^{ftop} of all finite topological covers of X is Galois with fundamental group $\widehat{\pi_1^{top}(X,x)}$.*

2.2 Etale covers

Let X be a connected, locally noetherian scheme and R_X^{et} the category of finite etale covers of X. For any geometric point $x : \mathrm{spec}(\Omega) \to X$, let $F_x : R_X^{et} \to FSets$, $f : Y \to X \mapsto Y_x(\Omega)$ denote the functor "geometric fiber over x". Then:

Theorem 2.6 *The category R_X^{et} is Galois with fibre functors F_x, $x \in X(\Omega)$, $\Omega = \overline{\Omega}$.*

Proof. See [Mur67, p. 54-63]. One has to check axioms (1) to (6) of the definition of a Galois category.
Axiom (1): R_X^{et} has a final object $Id_X : X \to X$ and, for any $f_i : Y_i \to X \in R_X^{et}$, $i = 1, 2$, one classically has $Y_1 \times_{p_1,X,p_2} Y_2 \to X \in R_X^{et}$.
Axiom (2): R_X^{et} has an initial object $\emptyset$ and, for any $f_i : Y_i \to X \in R_X^{et}$, $i = 1, 2$, one straightforwardly has that $Y_1 \coprod Y_2 \to X \in R_X^{et}$.

Lemma 2.7 *Universal quotients by finite groups exist in R_X^{et}. Furthermore, such quotients are strict epimorphisms in R_X^{et}.*

Proof. Let $f : Y \to X \in R_X^{et}$ and let G be a finite group such that $\alpha : G \to \mathrm{Aut}_{R_X^{et}}(f)$. We have to show that there exists $\overline{f} : Y/G \to X \in R_X^{et}$ and $\pi \in \mathrm{Hom}_{R_X^{et}}(f, \overline{f})$ such that for any $f' : Y' \to X$ and for any $\pi' \in \mathrm{Hom}_{R_X^{et}}(f, f')$ with $\pi' \circ \alpha(g)(f) = \pi' \circ f$, $g \in G$ there exists a unique $\overline{\pi}' \in \mathrm{Hom}_{R_X^{et}}(\overline{f}, f')$ satisfying $\overline{\pi}' \circ \pi = \pi'$.

1. $Y = \mathrm{spec}(A)$, $X = \mathrm{spec}(B)$: Then one straightforwardly checks that $\overline{f} : \overline{Y} := \mathrm{spec}(A^G) \to X$ is the universal quotient of $f : Y \to X$ in the category of X-schemes. It remains to prove that $\overline{f} \in R_X^{et}$. Since $f : Y \to X$ is finite, so is $\overline{f} : \mathrm{spec}(A^G) \to X$ it actually only remains to prove that $\overline{f} : \overline{X} := \mathrm{spec}(A^G) \to X$ is etale.

(a) Let $X' = \mathrm{spec}(B') \to X$ be a flat, affine base change and consider the following cartesian diagram:

$$\begin{tikzcd}
Y' \arrow[r, "\pi'"] \arrow[d] \arrow[rr, bend left, "f'"] \arrow[dr, phantom, "\square"] & \overline{X}' \arrow[r, "\overline{f}'"] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\
Y \arrow[r, "\pi"'] \arrow[rr, bend right, "f"'] & \overline{X}' \arrow[r, "\overline{f}"'] & X.
\end{tikzcd}$$

Let $\alpha' : G \to \mathrm{Aut}_{R^{et}_{X'}}(f')$ denote the canonical induced action of G on Y'. Then $\overline{f}' : \overline{Y}' \to X'$ *is the universal quotient of* $f' : Y' \to X'$ *in the category of* X'*-schemes that is,* $Y' = \mathrm{spec}((A \otimes_B B')^G)$.
Indeed, one has the exact sequence of B-algebras:

$$0 \to A^G \to A \overset{\sum_{g\in G}(Id_A - g\cdot)}{\to} \bigoplus_{g\in G} A$$

Hence, since $B \to B'$ is a flat B-algebra, one gets the exact sequence of B'-algebras

$$0 \to A^G \otimes_B B' \to A \otimes_B B' \overset{\sum_{g\in G}(Id_A - g\cdot)\otimes_B Id_{B'}}{\to} \bigoplus_{g\in G} A \otimes_B B',$$

whence $A^G \otimes_B B' = (A \otimes_B B')^G$.

(b) Let $\overline{y} \in \overline{Y}$. Then $X' := \mathrm{spec}(\mathcal{O}_{X,\overline{f}(\overline{y})}) \to X$ is an affine flat base change as above and there is a unique $\overline{y}' \in \overline{Y}'$ lying above $\overline{y}$ in $\overline{Y}'$. Furthermore, $\mathcal{O}_{\overline{Y},\overline{y}} = \mathcal{O}_{\overline{Y}',\overline{y}'}$. In particular, $\overline{f} : \overline{X} \to X$ is etale at $\overline{y} \in \overline{Y}$ if and only if $\overline{f}' : \overline{X}' \to X'$ is etale at $\overline{y}' \in \overline{Y}'$. As a result, *one may assume that* B *is a local noetherian ring with maximal ideal say* $\mathcal{M}_B$.

(c) It also follows from faithfully flat descent that for any faithfully flat morphism $X' \to X$, $\overline{f} : \overline{Y} \to X$ is etale if and only if $\overline{f}' : \overline{Y}' \to X'$ is etale. Since $\mathrm{spec}(\widehat{B}) \to \mathrm{spec}(B)$ is faithfully flat, where $\widehat{B}$ denotes the completion of B with respect to its maximal ideal $\mathcal{M}_B$, *one may assume that* B *is a complete local noetherian ring.*

(d) Let $x \in X$ be the closed point of X. Since $f : Y \to X \in R^{et}_X$, for any $y \in f^{-1}(x)$, $k(y)/k(x)$ is a finite separable extension so one can choose a finite Galois extension $K/k(x)$ such that $k(y) \hookrightarrow K$, $y \in f^{-1}(x)$. Then there exists a flat local finite B-algebra $B \to B'$ such that $B'/\mathcal{M}_{B'} = K$ [EGA3, Prop. 10.3.1, Cor. 10.3.2]. Thus, by step (b) above, *one may assume that* $k(y) = k(x)$, $y \in f^{-1}(x)$.

(e) Now, one has:

$$A = \bigoplus_{y\in f^{-1}(x)} B = \bigoplus_{O\in f^{-1}(x)/G} \bigoplus_{y\in O} B.$$

Hence:

$$A^G = \bigoplus_{O\in f^{-1}(x)/G} (\bigoplus_{y\in O} B)^G = \bigoplus_{O\in f^{-1}(x)/G} B,$$

Whence the conclusion.

2. <u>General case:</u> Reduce to case 1. by using that $f : Y \to X \in R^{et}_X$ is finite hence affine (local existence) and the unicity of universal quotient up to canonical isomorphism (glueing).

Eventually, observe that $\pi : Y \to \overline{Y} \in R^{et}_{\overline{Y}}$ hence is open so $\pi(Y)$ is an open subscheme of $\overline{Y}$ coinciding with the scheme-theoretic image of π. If $\pi(Y) \neq \overline{Y}$ then, by the universal property of the scheme-theoretic image $\pi(Y) \to X$ would satisfy, as well, the universal property of quotient in R^{et}_X: a contradiction. So π is faithfully flat hence a strict epimorphism in R^{et}_X. $\square$

<u>Axiom (3)</u>: For any $f_i : Y_i \to X \in R^{et}_X$, $i = 1, 2$ and for any $u \in \mathrm{Hom}_{R^{et}_X}(f_1, f_2)$, $u : Y_1 \to Y_2 \in R^{et}_{Y_2}$ hence is both open and closed. In particular, with $Y'_2 := u(Y_1)$, $Y''_2 := Y_2 \setminus Y'_2$, one has $Y_2 = Y'_2 \coprod Y''_2$ and u factors as $u : Y_1 \stackrel{u|^{Y'_2}=U'}{\to} Y'_2 \stackrel{i_{Y'_2}=u''}{\to} Y_2 = Y'_2 \coprod Y''_2$ with u' a faithfully flat morphism hence a strict epimorphism in R^{et}_X and u'' an open immersion hence a monomorphism in R^{et}_X.

<u>Axiom (4)</u>: Just observe that $F_x(f : Y \to X) = \emptyset$ if and only if $Y = \emptyset$ and that F_x commutes with fibered products.
<u>Axiom (5)</u>: The fact that F_x commutes with direct sums and transforms strict epimorphisms into strict epimorphisms is straightforward. So it only remains to prove that F_x commutes with universal quotients by finite groups of automorphisms.
So, let $f : Y \to X \in R^{et}_X$ and let G be a finite group such that $\alpha : G \to \mathrm{Aut}_{R^{et}_X}(f)$. By functoriality, one gets $\alpha : G \to \mathrm{Aut}_{FSets}(F_x(f))$ and, since $\pi : f \to \overline{f}$ is the universal quotient of f by G in R^{et}_X, one has (i) $F_x(\pi) : F_x(f) \twoheadrightarrow F_x(\overline{f})$ is surjective and (ii) for any $g \in G$ $F_x(\pi) \circ \alpha(g) = F_x(\pi)$ hence $F_x(\pi) : F_x(f) \twoheadrightarrow F_x(\overline{f})$ factors canonically through $F_x(f)/G \twoheadrightarrow F_x(\overline{f})$. And, actually, $F_x(f)/G \tilde{\to} F_x(\overline{f})$ is an isomorphism. Indeed, this follows from:

Lemma 2.8 *Let $f : Y \to X \in R^{et}_X$ and let G be a finite group such that $\alpha : G \to \mathrm{Aut}_{R^{et}_X}(f)$. Then:*
(i) G acts transitively on the fibers of $\pi : Y \to Y/G \in R^{et}_{Y/G}$;
(ii) For any $y \in Y$, set $D_\pi(y) := \mathrm{Stab}_G(y) \subset G$ for the decomposition group of y. Then $k(y)/k(\pi(y))$ is a Galois extension and the canonical morphism $D_\pi(y) \twoheadrightarrow \mathrm{Gal}(k(y)|k(\pi(y)))$ is an epimorphism.

Proof. As in the proof of lemma 2.7, one may assume that $Y = \mathrm{spec}(A)$, $Y = \mathrm{spec}(A^G)$ and $\overline{y} = \mathcal{M} \in \mathrm{spm}(A^G)$ is a closed point.
(i)Let $y_i = \mathcal{P}_i \in \pi^{-1}(\overline{y})$, $i = 1, 2$. Then, as $A^G \to A$ is a finite A^G-algebra, it follows from the going up theorem that $\mathcal{P}_1$, $\mathcal{P}_2$ are also maximal ideals. Assume that $\mathcal{P}_1 \neq g\mathcal{P}_2$, $g \in G$ then, by the Chinese remainder theorem, there exists $a_1 \in \mathcal{P}_1 \setminus \cup_{g \in G} g\mathcal{P}_2$. Hence $\prod_{g \in G} ga_1 \in A^G \cap \mathcal{P}_1 \setminus A^G \cap \mathcal{P}_2$: a contradiction.
(ii) Let $\mathcal{P} \in \pi^{-1}(\overline{y})$. Then $k(y)/k(\overline{y})$ is a finite separable extension so there exists $a \in A$ such that $k(\overline{y})(\overline{a}) = k(y)$, where $\overline{a}$ denotes the reduction of $a \in A$ modulo $\mathcal{P}$. The polynomial $P_a := \prod_{g \in G}(T - ga) \in A^G[T]$ splits completely over A and its reduction $\overline{P}_a \in k(\overline{y})[T]$ modulo $\mathcal{P} \cap A^G$ splits completely over $k(y)$ and has root $\overline{a}$ hence $k(y) = k(\overline{y})(\overline{a})$ is normal over $k(\overline{y})$.

By definition of $D_\pi(y)$, $\mathcal{P} \neq g\mathcal{P}$, $g \notin D_\pi(y)$ hence, by the Chines remainder theorem, there exists $a_1 \in A$ such that $a_1 \equiv a \mod \mathcal{P}$ and $a_1 \equiv 0 \mod g^{-1}\mathcal{P}$, $g \notin D_\pi(y)$. By construction $k(y) = k(\overline{y})(\overline{a}) = k(\overline{y})(\overline{a}_1)$. Also, $\overline{P}_{a_1} \in k(\overline{y})[T]$ has roots $\overline{a}_1$ hence, for any $\sigma \in \mathrm{Gal}(k(y)|k(\overline{y}))$, $\sigma(\overline{a}_1)$ is again a root of $\overline{P}_{a_1}$ *i.e.* there exists $g_\sigma \in G$ such that $\overline{g_\sigma a_1} = \sigma(\overline{a}_1)$. But, by construction of a_1, $\overline{ga_1} = 0$, $g \notin D_\pi(y)$ whereas $\overline{a}_1 \neq 0$ hence $\sigma(\overline{a}_1) \neq 0$, which shows that $g_\sigma \in D_\pi(y)$ $\square$
<u>Axiom (6)</u>: For any $f_i : Y_i \to X \in R^{et}_X$, $i = 1, 2$ let $u \in \mathrm{Hom}_{R^{et}_X}(f_1, f_2)$ such that $F_x(u) : F_x(f_1) \tilde{\to} F_x(f_2)$ is bijective. Recall that $u : Y_1 \to Y_2 \in R^{et}_{Y_2}$ hence the above hypothesis shows that $u : Y_1 \to Y_2 \in R^{et}_{Y_2}$

has rank 1 hence is an isomorphism. $\square$

For any geometric point $x : \mathrm{spec}(\Omega) \to X$,

$$\pi_1(X, x) := \pi_1(R_X^{et}; F_x)$$

is the *etale fundamental group of X with base point x*. Similarly, for any two geometric points $x_i : \mathrm{spec}(\Omega_i) \to X$, $i = 1, 2$,

$$\pi_1(X; x_1, x_2) := \pi_1(R_X^{et}; F_{x_1}), F_{x_2})$$

is the set of *etale paths from x_1 to x_2*. (Note that Ω_1 and Ω_2 may have different characteristics).

It follows from theorem 1.3 that $\pi_1(X; x_1, x_2) \neq \emptyset$ and that $\pi_1(X, x_1)\tilde{\to}\pi_1(X, x_2)$ canonically, up to inner automorphisms.

Eventually, given a morphism $\phi : X' \to X$ of connected, locally noetherian schemes and a geometric point $x' : \mathrm{spec}(\Omega) \to X'$, the base change functor $H(\phi) : R_X^{et} \to R_{X'}^{et}$, $f : Y \to X \mapsto p_2 : Y \times_{f,X,\phi} X' \to X'$ satisfies:

$$F_{x'} \circ H(\phi)(f) = (Y \times_{f,X,\phi} X')_{x'}(\Omega) \overset{(*)}{=} Y_x(\Omega) = F_{\phi(x')}(f),$$

where the equality $(*)$ follows from the universal property of fibre product. Hence $H(\phi) : R_X^{et} \to R_{X'}^{et}$ is a fundamental functor and one gets, correspondingly, a canonical profinite group morphism:

$$\pi_1(\phi) : \pi_1(X', x') \to \pi_1(X, \phi(x)).$$

Note that if $\phi : X' \to X \in R_X^{et}$ then $\pi_1(\phi) : \pi_1(X', x') \hookrightarrow \pi_1(X, \phi(x))$ is a monomorphism with image $\mathrm{Stab}_{\pi_1(X,\phi(x))}(x')$.

2.2.1 Spectrum of a field

Let k be a field and set $X := \mathrm{spec}(k)$. Then:

Proposition 2.9 *For any geometric point $x : \mathrm{spec}(\Omega) \to X$, there is a profinite group isomorphism:*

$$c_x : \pi_1(X, x)\tilde{\to}\Gamma_k,$$

canonical up to inner automorphisms.

Proof.

1. Consider the canonical diagram of schemes:

$$\begin{tikzcd} X & \mathrm{spec}(\Omega) \arrow[l, "x"'] \arrow[d] \\ \mathrm{spec}(k^s) \arrow[u, "x^s"] & \mathrm{spec}(\overline{k}). \arrow[l] \arrow[ul, "\overline{x}"'] \end{tikzcd}$$

   With these notation, it follows from theorem 1.3 that $\pi_1(X, x)\tilde{\to}\pi_1(X, \overline{x})$, canonically up to inner automorphisms. Hence one can assume that $\Omega = \overline{k}$.

2. By definition of the etale fundamental group, one has:

$$\begin{aligned}\pi_1(X,x) &= \mathrm{Aut}_{Fct}(\mathrm{Hom}_{Sch/k}(\mathrm{spec}(\overline{k}),-))\\ &\overset{(*)}{=} \mathrm{Aut}_{Sch/k}(\mathrm{spec}(\overline{k}))\\ &= \mathrm{Aut}(\overline{k}|k)\\ &\overset{(**)}{=} \Gamma_k,\end{aligned}$$

where the equality $(*)$ follows from Yoneda lemma and the equality $(**)$ is the canonical restriction-to-k^s isomorphism. $\square$

2.2.2 Normal base scheme

Fix an integral scheme X with function field $k(X)$. For any $f: Y \to X \in FR_X$ let $R(Y)$ denote the *ring of rational functions on* Y *i.e.* the direct product of the local rings of the generic points of Y; it comes equipped with a natural structure $R(f): k(X) \hookrightarrow R(Y)$ of $k(X)$-algebra and this defines a functor $R: FR_X \to FSA_{k(X)}$.

Given a finite field extension $i: k(X) \hookrightarrow L$, recall that the *normalization* $\pi_i: \tilde{X}^i \to X$ *of* X *in* $i: k(X) \hookrightarrow L$ is the solution of the following universal problem. For any dominant normal X-scheme $f: Y \to X$ such that $R(f): k(X) \hookrightarrow R(Y)$ factors through:

$$\begin{tikzcd} R(Y) & \\ k(X) \arrow[u, hook, "R(\pi)"] \arrow[r, hook, "i"'] & L \arrow[ul, hook, "R(f)_i"'] \end{tikzcd}$$

there exists a unique X-morphism $f_i: Y \to \tilde{X}^i$ such that $R(f_i) = R(f)_i$.

The normalization $\pi_i: \tilde{X}^i \to X$ always exists and is unique (up to a unique X-isomorphism). Furthermore, for any affine open subscheme $U \subset X$, $\pi_i^{-1}(U) \subset \tilde{X}^i$ is again an open affine subscheme and the corresponding ring extension $\pi_i^{\#}(U): \mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{\tilde{X}^i}(\pi_i^{-1}(U))$ is the integral closure of $\mathcal{O}_X(U)$ in L. In particular, $\tilde{X}^i$ is normal. When X is normal, $\pi_i: \tilde{X}^i \to X$ is a finite morphism. From now on, we assume that X is also normal.

Given a finite separable $k(X)$-algebra $i: k(X) \hookrightarrow A = \prod_{j=1}^r L_j$, let $i_j: k(X) \hookrightarrow L_j$ denote the composition of $i: k(X) \hookrightarrow A$ with the jth projection $p_j: A \twoheadrightarrow L_j$, $j = 1, ..., r$. Also, define the normalization of X in $i: k(X) \hookrightarrow A$ to be the coproduct $\coprod_{j=1}^r \pi_{i_j}: \coprod_{j=1}^r \tilde{X}^{i_j} \to X$ and denote it, again, by $\pi_i: \tilde{X}^i \to X$. Then a finite separable $k(X)$-algebra $i: k(X) \hookrightarrow A = \prod_{j=1}^r L_j$ is *etale over* X if $\pi_i: \tilde{X}^i \to X$ is unramified (or, equivalently, etale). We denote by $FEA_{k(X),X}$ the category of finite separable $k(X)$-algebras etale over X.

Theorem 2.10 *Let* X *be a connected normal scheme. The function ring functor induces an equivalence of categories* $R: R_X^{et} \approx FEA_{k(X),X}$ *a pseudo-inverse of which is given by the normalization functor* $\pi_-: FEA_{k(X),X} \approx R_X$, $i: k(X) \hookrightarrow A \mapsto \pi_i: \tilde{X}^i \to X$.

Proof. See [SGA1, Chap. I, §10]. $\square$

Corollary 2.11 *Let X be a connected, locally noetherian, normal scheme with generic point η : $k(X) \to X$. Let $k(X) \hookrightarrow \Omega$ be an algebraically closed field extension defining geometric points x_η : $\mathrm{spec}(\Omega) \to \mathrm{spec}(k(X))$ and $x : \mathrm{spec}(\Omega) \to X$. Let $k(X) \hookrightarrow M_{k(X),X}$ denote the maximal algebraic field extension of $k(X)$ in Ω which is etale over X.Then one has the canonical short exact sequence of profinite groups:*

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_{M_{k(X),X}} \arrow[rr] & & \Gamma_{k(X)} \arrow[d, equal] & & \\
& & \pi_1(\mathrm{spec}(k(X)), x_\eta) \arrow[r] & \pi_1(X, x) \arrow[r] & 1.
\end{tikzcd}$$

In particular, this defines a canonical profinite group isomorphism:

$$\mathrm{Gal}(M_{k(X),X}|k(X)) \tilde{\to} \pi_1(X, x).$$

Proof. From theorem 2.10, the base change functor $H(\eta) : R_X^{et} \to R_{\mathrm{spec}(k(X))}^{et}$ is nothing but the forgetful functor $For : FEA_{k(X),X} \hookrightarrow FSA_{k(X)}$. By the definition of the category $FEA_{k(X),X}$, the natural functor morphism $\mathrm{Hom}_{FEA_{k(X),X}}(-, M_{k(X),X}) \hookrightarrow \mathrm{Hom}_{FEA_{k(X),X}}(-, \Omega)$ induced by the inclusion $M_{k(X),X} \hookrightarrow \Omega$ is a functor isomorphism. Hence, $\mathrm{Hom}_{FEA_{k(X),X}}(-, M_{k(X),X}) : FEA_{k(X),X} \to FSets$ is also a fibre functor for $FEA_{k(X),X}$. Also, from §2.2.1, we may assume that $\Omega = \overline{k(X)}$. Then:

$$u_{H(\eta)} : \pi_1(\mathrm{Spec}(k(X)), x_\eta) \to \pi_1(X, x)$$

corresponds to the natural functor morphism:

$$\mathrm{Aut}_{Fct}(\mathrm{Hom}_{FSA_{k(X)}}(-, \overline{k(X)})) \to \mathrm{Aut}_{Fct}(\mathrm{Hom}_{FEA_{k(X),X}}(-, M_{k(X),X}))$$

which, by Yoneda lemma, identifies with:

$$\mathrm{Aut}_{FSA_{k(X)}}(\overline{k(X)}) \to \mathrm{Aut}_{FEA_{k(X),X}}(M_{k(X),X})$$

i.e. the restriction epimorphism $\mathrm{Aut}(\overline{k(X)}|k(X)) \twoheadrightarrow \mathrm{Gal}(M_{k(X),X}|k(X))$. Conclude again using the canonical restriction isomorphism $\mathrm{Aut}(\overline{k(X)}|k(X)) \tilde{\to} \Gamma_{k(X)}$. □

Example 2.12 Let X be a curve, smooth and geometrically connected over a field k and let $X \hookrightarrow \tilde{X}$ be the smooth compactification of X. Write $\tilde{X} \setminus X = \{P_1, \ldots, P_r\}$. Then, with the notation of corollary 2.11, $k(X) \hookrightarrow M_{k(X),X}$ is just the maximal algebraic extension of $k(X)$ in Ω unramified outside the places $P_1, \ldots, P_r$.

2.2.3 Geometrically connected varieties

Let k be a perfect field and let X be a scheme geometrically connected and of finite type over k. Fix a geometric point $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{k}}$ with image $x : \mathrm{spec}(\Omega) \to X$ and $s : \mathrm{spec}(\Omega) \to \mathrm{spec}(k)$.

Proposition 2.13 *Then the structural morphism $X \to k$ induces a canonical short exact sequence of profinite groups:*

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, x) \to \pi_1(\mathrm{spec}(k), s) \to 1. \quad (2)$$

Remark 2.14 The statement of proposition 2.13 remains true without the assumtion that k is perfect. But, for this, one needs an additional descent argument (see Step 1, §3.2.2).

Example 2.15 Assume furthermore that X is normal. Then the assumption that X is geometrically connected over k is equivalent to the assumption that $\overline{k} \cap k(X) = k$ and, with the notation of §2.2.2, the short exact sequence (2) is just the one obtains from usual Galois theory:

$$1 \to \mathrm{Gal}(M_{\overline{k}(X), X_{\overline{k}}}) \to \mathrm{Gal}(M_{k(X),X}) \to \Gamma_k \to 1.$$

The proof in the general case is slightly more difficult.

Proof. We use the criteria of proposition 1.18.
Exactness on the right: It follows from the fact that X is geometrically connected over k, which implies that for any finite field extension $k \hookrightarrow l$, X_l is connected.
Exactness on the left: This amounts to showing that for any $\overline{f} : \overline{Y} \to X_{\overline{k}} \in R^{et}_{X_{\overline{k}}}$ there exists $f : Y \to X \in R^{et}_X$ and a morphism from $f \times_k \overline{k} : Y_{\overline{k}} \to X_{\overline{k}}$ to $\overline{f} : \overline{Y} \to X_{\overline{k}}$ in $R^{et}_{X_{\overline{k}}}$. So, let l be a field of definition for $\overline{f} : \overline{Y} \to X_{\overline{k}}$, that we may assume to be Galois and finite over k (here, we use the assumption that X is of finite type over k). Then the action of Γ_k over $\overline{f} : \overline{Y} \to X_{\overline{k}}$ factors through $\Gamma := \mathrm{Gal}(l|k)$ and it follows from Weil descent that the cover $\coprod_{\sigma \in \Gamma} {}^\sigma\overline{f} : \coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y} \to X_{\overline{k}}$ is defined over k. Indeed, for any $\tau \in \Gamma$, let $\phi_\tau : {}^\tau(\coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y}) \tilde{\to} \coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y}$ defined by $\phi_\tau({}^\tau({}^\sigma\overline{Y}) = {}^{\tau\sigma}\overline{Y}$ and $\phi_\tau : {}^\tau({}^\sigma\overline{Y}) \tilde{\to} {}^{\tau\sigma}\overline{Y}$ is the canonical l-isomorphism fitting in:

$$\begin{tikzcd}
{}^{\tau\sigma}\overline{Y} \arrow[dr, "\phi_\tau^{-1}"] \arrow[drrr] \arrow[ddr] & & & \\
& {}^\tau({}^\sigma\overline{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}^\sigma\overline{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Y} \arrow[d] \\
& l \arrow[r, "\tau"'] \arrow[rr, bend right, "\tau\sigma"'] & l \arrow[r, "\sigma"'] & l
\end{tikzcd}$$

Since $\phi_\sigma \circ {}^\sigma\phi_\tau = \phi_{\sigma\tau}$, $\sigma,\ \tau \in \Gamma$, the $\phi_\tau : {}^\tau(\coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y}) \tilde{\to} \coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y}$, $\tau \in \Gamma$ are a descent datum for $\coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y}$. Since $\overline{Y}$ is of finite type over k, one can cover it by finitely many open affine subschemes $U_i = \mathrm{spec}(A_i)$, $1 \le i \le r$ and, up to enlarging l, one may assume that each of the U_i is defined over l. But then

$$\coprod_{\sigma\in\Gamma} {}^\sigma\overline{Y} = \coprod_{\sigma\in\Gamma} {}^\sigma \bigcup_{1\le i\le r} {}^\sigma U_i = \bigcup_{1\le i \le r} \coprod_{\sigma\in\Gamma} {}^\sigma U_i,$$

where $\coprod_{\sigma\in\Gamma} {}^\sigma U_i = \mathrm{spec}(\prod_{\sigma\in\Gamma} {}^\sigma A_i)$ is an open affine subscheme Γ-stable. So the descent datum is effective, that is there exist a a k-scheme Y and a $\overline{k}$-isomorphism $\phi : \overline{Y} \tilde{\to} Y_{\overline{k}}$ (actually defined over l) such that $\phi_\tau = {}^\tau\phi \circ \phi^{-1}$, $\tau \in \Gamma$. Then, since *via* this identification $\coprod_{\sigma\in\Gamma} {}^\sigma\overline{f} : Y_{\overline{k}} \to X_{\overline{k}}$ commutes with the Γ-action, it follows from Weil descent for morphisms that $\coprod_{\sigma\in\Gamma} {}^\sigma\overline{f} : Y_{\overline{k}} \to X_{\overline{k}}$ is defined over k that is there exist a k-morphism $f : Y \to X$ such that $f \times_k \overline{k} : Y_{\overline{k}} \to X_{\overline{k}}$ is isomorphic to $\overline{f} : \overline{Y} \to X_{\overline{k}}$ in $R^{et}_{X_{\overline{k}}}$. Eventually, since $X_{\overline{k}} \to X$ is faithfully flat, it follows from faithfully flat descent that $f : Y \to X \in R^{et}_X$.
Exactness in the middle:
- For any connected $\phi : \mathrm{spec}(l) \to \mathrm{spec}(k) \in R^{et}_{\mathrm{spec}(k)}$, $(\phi \times_k X) \times_X X_{\overline{k}}$ is just the identity $X_{\overline{k}} \tilde{\to} X_{\overline{k}}$.
- For any connected $f : Y \to X \in R^{et}_X$ such that $f \times_k \overline{k} : Y_{\overline{k}} \to X_{\overline{k}}$ admits a section, say $s : X_{\overline{k}} \hookrightarrow Y_{\overline{k}}$, let l be a field of definition of $s : X_{\overline{k}} \hookrightarrow Y_{\overline{k}}$, that we may assume to be finite over k. Then there exists a section $s_l : X_l \hookrightarrow Y_l$ of $f_l : Y_l \to X_l$ (such that $s_l \times_l \overline{k} : (X_l)_{\overline{k}} \hookrightarrow (Y_l)_{\overline{k}}$ identifies with $s : X_{\overline{k}} \hookrightarrow Y_{\overline{k}}$) hence a morphism from $X_l \to X$ to $f : Y \to X$ (obtained by composing s with the cover $Y_l \to Y$. $\square$

Remark 2.16 Any k rational point on X produces a (conjugacy classe of) splitting(s) of (2). The converse question, that is whether any splitting of (2) comes from a k-rational point on X is what is at stake in the section conjecture. Also, determining group-theoretically which sections of (2) come from k rational points on X is often a crucial step in anabelian proofs. Note that the section conjecture is false in general (for instance, if $k = \mathbb{F}_p$ is finite then, in general, (2) admits infinitely many non conjugate splittings whereas $X(k)$ is finite).

2.2.4 Abelian varieties

For an introduction to the general theory of abelian varieties, we refer to [Mum70]. In the following, k always denotes an algebraically closed field.

Theorem 2.17 (Serre-Lang) *Let k be an algebraically closed field and A an abelian variety over k. For each $n \geq 1$ let $A[n]$ denote the finite subgroup underlying the kernel of the multiplication-by-n morphism $[n_A] : A \to A$ and, for each prime l, set*

$$T_l(A) := \varprojlim A[l^n]^3.$$

Then, there is a canonical isomorphism

$$\pi_1(A, 0_A) \tilde{\to} \prod_l T_l(A).$$

Proof (sketch of).

1. Claim: *Let $f : X \to A \in R_A^{et}$. Then X carries a structure of abelian variety such that $f : X \to A$ becomes a separable isogeny.*

   To prove this, we will use the following criterion [Mum70, App. to §4].

   Lemma 2.18 *Let X be an irreducible scheme proper and of finite type over k, $e \in X$ and $m : X \times_k X \to X$ a morphism of k-schemes such that $m(e, x) = m(x, e)$, $x \in X$. Then X is an abelian variety over k with group law m and identity e.*

   Let $\Gamma_m \hookrightarrow A \times_k A \times_k A$ denote the graph of the group law $m : A \times_k A \to A$ on A and consider the following cartesian square:

$$\begin{tikzcd} \Gamma_m \arrow[r, hook] & A \times_k A \times_k A \\ \Gamma' \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\square"] & X \times_k X \times_k X \arrow[u, "f \times f \times f"'] \end{tikzcd}$$

   Then, by construction, one has the following commutative square:

$$\begin{tikzcd} \Gamma' \arrow[r] \arrow[d, "p_{12}"'] & \Gamma_m \arrow[d, "p_{12}"'] \\ X \times_k X \arrow[r, "f \times f"'] & A \times_k A \end{tikzcd}$$

   and, on the one hand, since $f : X \to A$ is etale so are $f \times f : X \times_k X \times_k X \to A \times_k A$ and $\Gamma' \to \Gamma_m$ and, on the other hand, by definition of the graph, $p_{12} : \Gamma_m \tilde{\to} A \times_k A$ is a k-isomorphism hence

[3] Recall that if l is prime to the characteristic of k then $T_l(A) \simeq \mathbb{Z}_l^{2g}$ whereas if $l = p$ is the characteristic of k then $T_p(A) \simeq \mathbb{Z}_p^r$, where g and $r (\leq g)$ denotes the dimension and p-rank of A respectively.

is etale. So, it follows from the commutativity of the above diagram that $p_{12}:\Gamma'\to X\times_k X$ is etale.
Fix now $x_0\in X$ such that $f(x_0)=0_A$ and let Γ be the connected component of Γ' containing (x_0,x_0,x_0). Write $p:=p_{12}|_\Gamma:\Gamma\to X\times_k X$. The assertion of step 1 will then follow from lemma 2.18 and the following claim.

Claim: *$p:\Gamma\tilde{\to}X\times_k X$ is an isomorphism and, defining $m_X:X\times_k X\stackrel{p^{-1}}{\to}\Gamma\stackrel{p_3}{\to}X$, one has $m_X(x,x_0)=x=m_X(x_0,x)$, $x\in X$.*

It remains to prove the claim. First, as p is an etale cover it is enough to prove that there exists $(x_1,x_2)\in X\times_k X$ such that $|p^{-1}((x_1,x_2))|=1$. So let $\sigma_1,\ \sigma_2:X\to\Gamma$ defined by $\sigma_1(x)=(x_0,x,x)$ and $\sigma_2(x)=(x,x_0,x)$ (note that , for $i=1,\ 2$, $\sigma_i(X)$ is connected, $\sigma_i(X)\subset\Gamma'$ and $(x_0,x_0,x_0)=\sigma_i(x_0)\in\sigma_i(X)$ hence $\sigma_i(X)\subset\Gamma$). Since $p|_{\sigma_2(X)}:\sigma_2(X)\tilde{\to}X\times_k\{x_0\}$, it is enough to prove that $p^{-1}(X\times_k\{x_0\})=\sigma_2(X)$ or, equivalently, that $q^{-1}(x_0)=\sigma_2(X)$, where $q=p_2|_\Gamma:\Gamma\to X$. But as $\sigma_2(X)$ is an irreducible component of $q^{-1}(x_0)$, it is actually enough to prove that $q^{-1}(x_0)$ is irreducible. As already noticed, $p:\Gamma\to X\times_k X$ is etale and as $f\times f:X\times_k X\to A\times_k A$ is etale as well so is $\Gamma\to A\times_k A$ hence Γ is regular and, being connected, is irreducible. Furthermore $q=p_2\circ p:\Gamma\to X$ is smooth as the composite of the etale morphism p with the smooth morphism p_2 and $q\circ\sigma_1=Id_X$ so the claim follows from the following lemma [Mum70, Lemma p. 168].

Lemma 2.19 *Let S, T be irreducible schemes of finite over k and $f:S\to T$ a smooth proper k-morphism. If there exists a section $s:T\to S$ of f then all the fibres of f are irreducible.*

The last part of the claim follows then from $m_X(x,x_0)=p_3(p^{-1}(x,x_0))=p_3(\sigma_2(x))=x$ and $m_X(x_0,)=p_3(p^{-1}(x_0,x))=p_3(\sigma_1(x))=x$ (for the second equality, note that the parts of σ_1 and σ_2 can be interverted).

2. Now let $f:X\to A$ be an isogeny with kernel of exponent say $n\geq 1$. Then $\ker(f)\subset\ker([n_X])$ hence one has a canonical commutative diagram:

$$\begin{tikzcd} & X/\ker(f) \arrow[dl, "g"'] \arrow[r, equal] & A \\ X & X. \arrow[l, "{[n_X]}"] \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}$$

Also, it follows from the surjectivity of f that $f\circ g=[n_A]$.
Combining the above remark and step 1, one gets that $([l^n]:A\to A)_{n\geq 0}$ is cofinal among the finite etale covers of A with degree a power of l that is

$$\pi_1(A,0_A)^{(l)}=\varprojlim A[l^n]=T_l(A).\ \square$$

Remark 2.20 As already noticed, if k is any field and A is an abelian variety over k, there is always a canonical split short exact sequence of profinite groups:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(A\times_k\overline{k},0_A) \arrow[r] & \pi_1(A,0_A) \arrow[r] & \Gamma_k \arrow[r] \arrow[l, bend right, "0_A"'] & 1\ , \end{tikzcd}$$

which identifies canonically $\pi_1(A,0_A)$ with the Γ_k-module $\prod_l T_l(A)\rtimes\Gamma_k$.

Now, assume that $k = \mathbb{C}$ and that $A = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice. Then, on the one hand, the universal covering of A is just the quotient map $\mathbb{C}^g \to A$ and has group $\pi_1^{top}(A(\mathbb{C}), 0_A) \simeq \Lambda$ whereas, on the other hand, for any prime l:

$$\begin{aligned} T_l(A) &= \lim_{\longleftarrow} A[l^n] \\ &= \lim_{\longleftarrow} \frac{1}{l^n}\Lambda/\Lambda \\ &= \lim_{\longleftarrow} \Lambda/l^n\Lambda \\ &= \Lambda^{(l)}, \end{aligned}$$

whence

$$\pi_1(A, 0_A) \simeq \prod_l T_l(A) \simeq \prod_l \pi_1^{top}(A(\mathbb{C}), 0_A)^{(l)} \simeq \pi_1^{top}(\widehat{A(\mathbb{C})}, 0_A).$$

This is a special case of the more general Riemann existence theorem.

2.2.5 Riemann Existence Theorem

Complex analytic spaces

- *"Affine" complex analytic spaces.* Given analytic functions $f_1, \ldots, f_r : U \to \mathbb{C}$ defined on the polydisc $U \subset \mathbb{C}^n$ of all $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $|z_i| < 1$, $i = 1, \ldots, n$, let $\mathfrak{U}(f_1, \ldots, f_r)$ denote the locally ringed space in $\mathbb{C}$-algebra with:
- underlying topological space the closed subset $\cap_{i=1}^r f_i^{-1}(0) \subset U$ endowed with the topology inherited from the transcendent topology on U;
- structural sheaf $\mathcal{O}_U / < f_1, \ldots, f_r >$, where $\mathcal{O}_U$ is the sheaf of germs of holomorphic functions on U.

- *Complex analytic spaces.* The category $An_{\mathbb{C}}$ of complex analytic spaces is the full subcategory of the category $LR_{\mathbb{C}-Alg}$ of locally ringed spaces in $\mathbb{C}$-algebra whose objects $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ are locally isomorphic to affine complex analytic spaces.

Complex analytic spaces associated with a scheme locally of finite type over $\mathbb{C}$ Let X be a scheme locally of finite type over $\mathbb{C}$

Claim: *The functor* $\mathrm{Hom}_{LR_{\mathbb{C}-Alg}}(-, X) : An_{\mathbb{C}}^{op} \to Sets$ *is representable that is there exists a complex analytic space* X^{an} *and a morphism* $\phi_X : X^{an} \to X$ *in* $LR_{\mathbb{C}-Alg}$ *inducing a functor isomorphism*

$$\phi_X \circ : \mathrm{Hom}_{An_{\mathbb{C}}}(-, X^{an}) \tilde{\to} \mathrm{Hom}_{LR_{\mathbb{C}-Alg}}(-, X).$$

$\phi_X : X^{an} \to X$ is unique up to a unique X-isomorphism and is called the *complex analytic space associated with* X. It can be explicitly described as follows. Let $\{(U_i = \mathrm{spec}(A_i))_{i \in I}, (\phi_{i,j} : U_{i,j} \tilde{\to} U_{j,i})_{i,j \in I}\}$ be a glueing data for X by affine schemes. For each $i \in I$, since A_i is a $\mathbb{C}$-algebra of finite type, it can be written as $A_i = \mathbb{C}[\underline{X}] / < f_{i,1}, \ldots, f_{i,r_i} >$. Define X^{an} to be the complex analytic space given by the glueing data $\{(\mathfrak{U}(f_{i,1}, \ldots, f_{i,r_i}))_{i \in I}, (\phi_{i,j} : U_{i,j} \tilde{\to} U_{j,i})_{i,j \in I}\}$. For more details, see for instance [S56].

Eventually, given a $\mathbb{C}$-morphism $f : X \to Y$ of schemes locally of finite type over $\mathbb{C}$, it follows from the universal property of $\phi_Y : Y^{an} \to Y$ that there exists a unique morphism $f^{an} : X^{an} \to Y^{an}$ in $An_{\mathbb{C}}$ such that $\phi_Y \circ f^{an} = f \circ \phi_X$.

Statement Let $Sch^{LFT}/\mathbb{C}$ denote the category of schemes locally of finite type over $\mathbb{C}$. One thus gets a functor $(-)^{an} : Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}}$. Riemann existence theorem can now be formulated as follows.

Theorem 2.21 *For any scheme X locally of finite type over $\mathbb{C}$, the functor $(-)^{an} : Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}}$ induces an equivalence of categories*

$$(-)^{an} : R_X^{et} \approx R_{X^{an}}^{et}.$$

In particular, as $R_{X^{an}}^{et}$ is equivalent to the category $FR_{X^{top}}^{top}$ of finite topological covers of the underlying transcendent topological space X^{top} of X[4], for any $x \in X$ one has a canonical profinite groups isomorphism:

$$\widehat{\pi_1^{top}(X^{top}, x)} \simeq \pi_1(X, x).$$

Proof. See [SGA1, XII, Th. 5.1]. □

Example 2.22 Let X be a smooth connected $\mathbb{C}$-curve of type (g, r) (that is the projective compactification $\tilde{X}$ of X has genus g and $|\tilde{X} \setminus X| = r$). Then, for any $x \in X$ one has a canonical profinite group isomorphism $\widehat{\Gamma}_{g,r} \simeq \pi_1(X, x)$.

In particular, if $g = 0$ then $\pi_1(X, x)$ is the pro-free group on $r - 1$ generators, so, any finite group G generated by $\leq r-1$ elements is a quotient of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{t_1, \dots, t_r\}, x)$ or, equivalently, appears as the Galois group of a Galois extension $\mathbb{C}(T) \hookrightarrow K$ unramified everywhere except over $t_1, \dots, t_r$. This solves the inverse Galois problem over $\mathbb{C}(T)$.

3 Etale fundamental group

3.1 Descent

3.1.1 The formalism of descent

We recall briefly the formalism of descente. Let S be a scheme and $\mathcal{C}_S$ a subcategory of the category of S-schemes closed under fiber product. A *fibered category over $\mathcal{C}_S$* is a pseudofunctor $\mathfrak{X} : \mathcal{C}_S \to Cat$ that is the data of:
- for any $U \in \mathcal{C}_S$, a category $\mathfrak{X}_U$ (sometimes called the fibre of $\mathfrak{X}$ over $U \to S$);
- for any morphism $\phi : V \to U$ in $\mathcal{C}_S$, a base change functor $\phi^\star : \mathfrak{X}_U \to \mathfrak{X}_V$;
- for any morphisms $W \xrightarrow{\chi} V \xrightarrow{\phi} U$ in $\mathcal{C}_S$, a functor isomorphism $\alpha_{\chi,\phi} : \chi^\star \phi^\star \tilde{\to} (\phi \circ \chi)^\star$ satisfying the usual cocycle relations that is, for any morphisms $X \xrightarrow{\psi} W \xrightarrow{\chi} V \xrightarrow{\phi} U$ in $\mathcal{C}_S$, the following diagrams are commutative:

$$\begin{tikzcd} \psi^\star \chi^\star \phi^\star \arrow[r, "\psi^\star(\alpha_{\chi,\phi})"] \arrow[d, "\alpha_{\psi,\chi}(\phi^\star)"'] & \psi^\star(\phi \circ \chi)^\star \arrow[d, "\alpha_{\psi,\phi\circ\chi}"] \\ (\chi \circ \psi)^\star \phi^\star \arrow[r, "\alpha_{\chi\circ\psi,\phi}"'] & (\phi \circ \chi \circ \psi)^\star. \end{tikzcd}$$

Given a morphism $\phi : U' \to U$ in $\mathcal{C}_S$, write $U'' := U' \times_U U'$, $U''' := U' \times_U U' \times_U U'$, $p_i : U'' \to U'$, $i = 1, 2$, $p_{i,j} : U''' \to U''$, $1 \leq i < j \leq 3$, $u_i : U''' \to U'$, $i = 1,\ 2,\ 3$ for the canonical projections.

[4]To see this, observe that if $f : Y \to X^{top} \in FR_{X^{top}}^{top}$ is a topological covers then the local trivializations endows Y with a unique structure of analytic space (induced from X^{an}) and such that, with this structure, $f : Y \to X^{top}$ becomes an analytic cover. Conversely, if $f : Y \to X^{an} \in FR_{X^{an}}^{an}$ then, for any $y \in Y$ one can find open affine neighborhoods $V = \mathrm{spec}(B)$ of y and $U = \mathrm{spec}(A)$ of $f(y)$ such that $f(V) \subset U$, $B = A[X]/ < f >$ and $(\frac{\partial f}{\partial X})_y \in \mathcal{O}_{Y,y}^\times$ [L00, Prop. 4.11] hence the local inversion theorem gives local trivializations.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of descent for* $\mathfrak{X}$ if for any $x, y \in \mathfrak{X}_U$ and any morphism $f' : \phi^\star x \to \phi^\star y$ in $\mathfrak{X}_{U'}$ such that the following diagram commute:

$$\begin{tikzcd}
& p_1^\star\phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi(x)}"'] & p_1^\star y \arrow[dr, "\alpha_{p_1,\phi(y)}"] & \\
{\phi'}^\star(x) \arrow[rrr, dotted, "p_1^\star f'", "p_2^\star f'"'] \arrow[dr, "\alpha_{p_2,\phi(x)}"'] & & & {\phi'}^\star(y) \arrow[dl, "\alpha_{p_2,\phi(y)}"] \\
& p_2^\star\phi^\star(x) \arrow[r, "p_2^\star f'"] & p_1^\star y &
\end{tikzcd}$$

there exists a unique morphism $f : x \to y$ in $\mathfrak{X}_U$ such that $\phi^\star f = f'$.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of effective descent for* $\mathfrak{X}$ if $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ and if for any $x' \in \mathfrak{X}_{U'}$ and any isomorphism $u : p_1^\star(x') \tilde{\to} p_2^\star(x')$ in $\mathfrak{X}_{U''}$ such that the following diagram commute

$$\begin{tikzcd}
& p_{1,3}^\star p_1^\star(x') \arrow[r, "p_{1,3}^\star u"] \arrow[dl, "\alpha_{p_{1,3},p_1}(x')"'] & p_{1,3}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,3},p_2}(x')"] & \\
u_1^\star(x') \arrow[rrr, dotted, "p_{1,3}^\star u"] \arrow[dddr, dotted, "p_{1,2}^\star u"] & & & u_3^\star(x') \\
p_{1,2}^\star p_1^\star(x') \arrow[u, "\alpha_{p_{1,2},p_1}(x')"] \arrow[d, "p_{1,2}^\star u"'] & & & p_{2,3}^\star p_2^\star(x') \arrow[u, "\alpha_{p_{2,3},p_2}(x')"'] \\
p_{1,2}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,2},p_2}(x')"'] & & & p_{2,3}^\star p_1^\star(x') \arrow[u, "p_{2,3}^\star u"'] \arrow[dl, "\alpha_{p_{2,3},p_1}(x')"'] \\
& u_2^\star(x') \arrow[r, equal] & u_2^\star(x') \arrow[uuur, dotted, "p_{2,3}^\star u"] &
\end{tikzcd}$$

there is a (necessarily unique since $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$) $x \in \mathfrak{X}_U$ and an isomorphism $f' : \phi^\star(x) \tilde{\to} x'$ in $\mathfrak{X}_{U'}$ such that the following diagram commute

$$\begin{tikzcd}
& p_1^\star\phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi(x)}"'] & p_1^\star(x') \arrow[dd, "u"] \\
{\phi'}^\star(x) \arrow[urr, dotted, "p_1^\star f'"] \arrow[drr, dotted, "p_2^\star f'"] & & \\
& p_2^\star\phi^\star(x) \arrow[r, "p_2^\star f'"] \arrow[ul, "\alpha_{p_2,\phi(x)}"] & p_2^\star(x')
\end{tikzcd}$$

The pair $\{x',\ u : p_1^\star(x') \tilde{\to} p_2^\star(x')\}$ is called a descent datum for $\mathfrak{X}$ relatively to $\phi : U' \to U$. Denoting by $\mathfrak{D}(\phi)$ the category of descent data for $\mathfrak{X}$ relatively to $\phi : U' \to U$, saying that $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is fully faithfull and saying that $\phi : U' \to U$ is a morphism of effective descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is an equivalence of category.

Example 3.1 The basic example is that any faithfully flat and quasi-compact morphism $\phi : U' \to U$ is a morphism of effective descent for the fibered category of quasi-coherent modules. See for instance [FGA05, Part.1], for a comprehensive introduction to descent technics.

3.1.2 Selected results

The fibered categories we will now focus our attention on are the categories of finite etale covers. We will only mention results that will be used later. For the proofs, we refer to [SGA1, Chap. VIII and IX] .

Theorem 3.2 *Let X be a locally noetherian scheme and $i : X^{red} \hookrightarrow X$ be the underlying reduced closed subscheme. Then the functor $i^\star : R_X^{et} \approx R_{X^{red}}^{et}$ is an equivalence of categories. In particular, if X is connected, for any geometric point $x \in X^{red}$ one has a canonical profinite group isomorphism*

$$\pi_1(i) : \pi_1(X^{red}, x) \tilde{\to} \pi_1(X, x).$$

Theorem 3.3 *Let S be a locally noetherian scheme and let $f : S' \to S$ be a morphism which is either:*
- finite and surjective or
- faithfully flat and quasi-compact .
Then $f : S' \to S$ is a morphism of effective descent for the fibered category of etale, separated schemes of finite type.

Corollary 3.4 *Let S be a locally noetherian scheme and let $f : S' \to S$ be a morphism which is either:*
- finite, radiciel and surjective or
- faithfully flat, quasi-compact and radiciel.
Then $f : S' \to S$ induces an equivalence of categories $f^\star : R_S^{et} \approx R_{S'}^{et}$.

Theorem 3.5 *Let S be a locally noetherian scheme and let $f : S' \to S$ be a proper and surjective morphism. Then $f : S' \to S$ is a morphism of effective descent for the fibered category of etale covers.*

3.1.3 Comparison of fundamental groups for morphism of effective descent

Assume that $f : S' \to S$ is a morphism of effective descent for the fibered category of etale covers. Our aim is to interpret this in terms of fundamental groups.

Consider the usual notation S'', S''',

$$\begin{aligned} &p_i : S'' \to S',\ i = 1, 2,\\ &p_{i,j} : S''' \to S'',\ 1 \leq i < j \leq 3,\\ &u_i : S''' \to S',\ = 1, 2, 3. \end{aligned}$$

and assume that S, S', S'', S''' are disjoint union of connected schemes, then, with $E' := \pi_0(S')$, $E'' := \pi_0(S'')$, $E''' := \pi_0(S''')$, also set:

$$\begin{aligned} &q_i = \pi_0(p_i) : E'' \to E',\ i = 1, 2,\\ &q_{i,j} = \pi_0(p_{i,j}) : E''' \to E'',\ 1 \leq i < j \leq 3,\\ &v_i = \pi_0(u_i) : E''' \to E',\ i = 1, 2, 3. \end{aligned}$$

Write $\mathcal{C} := R_S^{et}$, $\mathcal{C}' := R_{S'}^{et}$, $\mathcal{C}'' := R_{S''}^{et}$, $\mathcal{C}''' := R_{S'''}^{et}$. We assume that S is connected.

Fix $s'_0 \in E'$ and for each $s' \in E'$, fix an element $\overline{s'} \in E''$ such that

$$q_1(\overline{s'}) = s'_0 \text{ and } q_2(\overline{s'}) = s'.$$

Also, for any $s' \in E'$ (resp. $s'' \in E''$, $s''' \in E'''$) fix a geometric point $\underline{s'} \in s'$ (resp. $\underline{s''} \in s''$, $\underline{s''} \in s''$) and write $\pi_{s'} := \mathrm{Aut}_{Fct}(F'_{\underline{s'}})$ (resp. $\pi_{s''} := \mathrm{Aut}_{Fct}(F''_{\underline{s''}})$, $\pi_{s'''} := \mathrm{Aut}_{Fct}(F'''_{\underline{s'''}})$) for the corresponding fundamental group.

Since for any $s'' \in E''$ $p_i(\underline{s''})$ and $\underline{q_i(s'')}$ lie in the same connected component of S', one gets etale paths $\alpha_i^{s''} : F''_{\underline{s''}} \circ p_i^\star = F'_{p_i(\underline{s''})} \tilde{\rightarrow} F'_{\underline{q_i(s'')}}$, hence profinite group morphisms:

$$q_i^{s''} : \pi_{s''} \rightarrow \pi_1(q_i(s''), p_i(\underline{s''})) \simeq \pi_{q_i(s'')},\ i = 1, 2.$$

Similarly, one gets etale paths $\alpha_{i,j}^{s'''} : F'''_{\underline{s'''}} \circ p_{i,j}^\star = F''_{p_{i,j}(\underline{s'''})} \tilde{\rightarrow} F''_{\underline{q_{i,j}(s''')}}$ and profinite group morphisms:

$$q_{i,j}^{s'''} : \pi_{s'''} \rightarrow \pi_1(q_{i,j}(s'''), p_i(\underline{s'''})) \simeq \pi_{q_{i,j}(s''')},\ 1 \leq i < j \leq 3.$$

Eventually, from the etale paths

$$\begin{array}{c}
F'''_{s'''} \circ p_{1,2}^\star \circ p_1^\star \tilde{\rightarrow} F_{\underline{v_1(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{1,3}^\star \circ p_1^\star; \\
F'''_{s'''} \circ p_{1,2}^\star \circ p_2^\star \tilde{\rightarrow} F_{\underline{v_2(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{2,3}^\star \circ p_1^\star; \\
F'''_{s'''} \circ p_{1,3}^\star \circ p_2^\star \tilde{\rightarrow} F_{\underline{v_3(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{2,3}^\star \circ p_2^\star;
\end{array}$$

one gets $a_i^{s'''} \in \pi_{v_i(s''')}$, $i = 1, 2, 3$ such that

$$\begin{array}{c}
q_1^{q_{1,2}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_1^{s'''}) \circ q_1^{q_{1,3}(s''')} \circ q_{1,3}^{s'''}; \\
q_2^{q_{1,2}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_2^{s'''}) \circ q_1^{q_{2,3}(s''')} \circ q_{2,3}^{s'''}; \\
q_2^{q_{1,3}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_3^{s'''}) \circ q_2^{q_{2,3}(s''')} \circ q_{2,3}^{s'''};
\end{array}$$

Since $f : S' \rightarrow S$ is a morphism of effective descent, the above data allows us to recover $\mathcal{C}$ from $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$ up to an equivalence of category hence to reconstruct $\pi_1(S, \underline{p(s'_0)})$ from the $\pi_{s'}$, $\pi_{s''}$, $\pi_{s'''}$.

More precisely, the category $\mathcal{C}'$ with descent data for $f : S' \rightarrow S$ is equivalent to the category $\mathcal{C}(\{\pi_{s'}\}_{s' \in E'})$ together with a collection of functor automorphisms $g_{s''} : Id \tilde{\rightarrow} Id$, $s'' \in E''$ satisfying the following relations:

(1) $g_{s''} q_1^{s''}(\gamma'') = q_1^{s''}(\gamma'') g_{s''}$, $s'' \in E''$;
(2) $g_{\overline{s'}} = g_{\overline{s'_0}}$, $s' \in E'$;
(3) $a_3^{s'''} g_{q_{1,3}(s''')} a_1^{s'''} = g_{q_{2,3}(s''')} a_2^{s'''} g_{q_{1,2}(s''')}$, $s''' \in E'''$,

So, set

$$\Phi := \coprod_{s' \in S'} \pi_{s'} \coprod_{s'' \in E''} \hat{\mathbb{Z}} g_{s''} / < (1), (2), (3) >,$$

where $\coprod$ stands for the free product in the category of profinite groups and let $\mathcal{N}$ be the class of all normal subgroups $N \lhd \Phi$ such that $[\Phi : N]$ and $[\pi_{s'} : i_{s'}^{-1}(N)]$ are finite (here $i_s : \pi_s \hookrightarrow \coprod_{s' \in S'} \pi_{s'} \coprod_{s'' \in E''} \hat{\mathbb{Z}} g_{s''} \twoheadrightarrow \Phi$ denotes the canonical morphism). Then writing

$$\pi := \varprojlim_{N \in \mathcal{N}} \Phi / N$$

one gets that the category $\mathcal{C}'$ with descent data for $f : S' \to S$ is also equivalent to the category $\mathcal{C}(\pi)$. Whence:

Theorem 3.6 *With the above assumptions and notation, one has a canonical profinite group isomorphism*

$$\pi_1(S, p(s'_0)) \tilde{\to} \pi.$$

Corollary 3.7 *With the above assumptions and notation, if E' and E'' are finite and if the $\pi_{s'}$, $s' \in E'$ are topologically of finite type then so is $\pi_1(S, p(s'_0))$.*

Corollary 3.8 *Let S be a connected scheme and let $f : S' \to S$ be a universally submersive and geometrically connected morphism. Then S' is connected and for any geometric point $s' \in S'$ the canonical profinite group morphism $\pi_1(f^\star) : \pi_1(S', s') \twoheadrightarrow \pi_1(S, f(s'))$ is an epimorphism.*

If, furthermore, $f : S' \to S$ is a morphism of effective descent for the fibered category of etale covers, let $s'' := (s', s') \in S''$ and $\pi_1(p_i^\star) : \pi_1(S'', s'') \to \pi_1(S', s')$, $i = 1,\ 2$ the two canonical profinite group morphisms induced by the canonical projections $p_i : S'' \to S'$, $i = 1, 2$. Then

$$\ker(\pi_1(f^\star)) = \overline{\mathrm{Nor}_{\pi_1(S',s')}(\{\pi_1(p_1^\star)(\gamma'')\pi_1(p_2^\star)(\gamma'')^{-1}\}_{\gamma'' \in \pi_1(S'',s'')})}.$$

3.2 Specialization

3.2.1 Statements

Let S be a locally noetherian scheme and $f : X \to S$ a proper, geometrically connected morphism with $f_* \mathcal{O}_X = \mathcal{O}_S$. Fix $s_0,\ s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$ and consider the following notation:

$$\begin{tikzcd}
\Omega_1 \arrow[d, "\overline{x}_1"'] \arrow[dr, "x_1"'] \arrow[drr, "x_{(1)}"] & & & & \Omega_0 \arrow[d, "\overline{x}_0"] \arrow[dl, "x_0"] \arrow[dll, "x_{(0)}"'] \\
\overline{X}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[dr, phantom, "\square"] & X_0 \arrow[l] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{X}_0 \arrow[l] \arrow[d] \\
\overline{k(s_1)} \arrow[r] & k(s_1) \arrow[r, "s_1"'] & S & k(s_0) \arrow[l, "s_0"] & \overline{k(s_0)}, \arrow[l]
\end{tikzcd}$$

where $\Omega_0,\ \Omega_1$ are algebraically closed fields. Also let $\overline{s}_0,\ \overline{s}_1$ denote the images of $\overline{x}_0,\ \overline{x}_1$ in S respectively.

The theory of specialization of fundamental groups consists, essentially, in comparing $\pi_1(\overline{X}_1, \overline{x}_1)$ and $\pi_1(\overline{X}_0, \overline{x}_0)$. The main result is the following.

Theorem 3.9 (Semi-continuity of fundamental groups) *There exists a morphism of profinite groups*

$$sp : \pi_1(\overline{X}_1, \overline{x}_1) \to \pi_1(\overline{X}_0, \overline{x}_0),$$

canonically defined up to inner automorphisms of $\pi_1(\overline{X}_0, \overline{x}_0)$. If, furthermore, $f : X \to S$ is separable, then $sp : \pi_1(\overline{X}_1, \overline{x}_1) \twoheadrightarrow \pi_1(\overline{X}_0, \overline{x}_0)$ is an epimorphism.

The morphism $sp : \pi_1(\overline{X}_1, \overline{x}_1) \to \pi_1(\overline{X}_0, \overline{x}_0)$ is called the *specialization morphism from s_1 to s_0*.

The proof of theorem 3.9 relies on the two following theorems. Let assume for a while that $S = \mathrm{Spec}(A)$ with A a local complete noetherian ring and that $s_0 \in S$ is the closed point of S, $s_1 \in S$ is any point of S.

Theorem 3.10 (First homotopy sequence) *The canonical sequence of profinite groups:*

$$1 \to \pi_1(\overline{X}_0, \overline{x}_0) \xrightarrow{i_0} \pi_1(X, x_{(0)})) \xrightarrow{p_0} \pi_1(S, \overline{s}_0) \to 1 \tag{3}$$

is exact and the canonical morphism $\Gamma_{k(s_0)} \tilde{\to} \pi_1(S, \overline{s}_0)$ *is an isomorphism.*

If, furthermore, $x_0 \in X(k(s_0))$ *(or is rational over a radiciel extension of* $k(s_0)$*) then the above short exact sequence splits.*

Theorem 3.11 (Second homotopy sequence) *Consider the following canonical sequence of profinite groups:*

$$\pi_1(\overline{X}_1, \overline{x}_1) \xrightarrow{i_1} \pi_1(X, x_{(1)}) \xrightarrow{p_1} \pi_1(S, \overline{s}_1). \tag{4}$$

Then $p_1 : \pi_1(X, x_{(1)}) \twoheadrightarrow \pi_1(S, \overline{s}_1)$ *is an epimorphism and* $\mathrm{Im}(i_1) \subset \mathrm{Ker}(p_1)$. *If, furthermore,* $f : X \to S$ *is separable then* $\mathrm{Im}(i_1) = \mathrm{Ker}(p_1)$.

Corollary 3.12 (Product) *Let* k *be an algebraically closed field,* $X \to k$ *a connected, proper* k*-scheme and* $Y \to k$ *a connected, locally noetherian* k*-scheme. Let* $x : \Omega \to X$ *and* $y : \Omega \to Y$ *be geometric points. Then the canonical profinite group morphism*

$$\pi_1(X \times_k Y, (x, y)) \tilde{\to} \pi_1(X, x) \times \pi_1(Y, y)$$

is an isomorphism.

In particular, if $Y = y : \mathrm{spec}(\Omega) \to \mathrm{spec}(k)$ then $\pi_1(X \times_{k,y} \Omega, x) \tilde{\to} \pi_1(X, x)$ is an isomorphism. In other words, the etale fundamental group of a connected, proper k-scheme is invariant under base extension by algebraically closed fields.

This is no longer true for non-proper schemes. Indeed, let k be an algebraically closed field of characteristic $p > 0$. From the long cohomology exact sequence associated with Artin-Schreier short exact sequence:

$$0 \to \mathbb{F}_p \to \mathbb{G}_{a,k} \xrightarrow{\mathcal{P}} \mathbb{G}_{a,k} \to 0$$

one gets:

$$k[T]/\mathcal{P}k[T] = \mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})/\mathcal{P}\mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \tilde{\to} \mathrm{H}^1_{et}(\mathbb{A}^1_k, \mathbb{F}_p) = \mathrm{Hom}_{ProGr}(\pi_1(\mathbb{A}^1_k, 0), \mathbb{F}_p).$$

An additive section of the canonical epimorphism $k[T] \twoheadrightarrow k[T]/\mathcal{P}k[T]$ is given by the representatives:

$$\sum_{n>0,(n,p)=1} a_n T^n, \ a_n \in k,$$

which shows that $\pi_1(\mathbb{A}^1_k, 0)$ is not of finite type (compare with theorem 3.22) and depends on the base field k.

Construction of the specialization morphism-1. Let assume, again, that $S = \mathrm{Spec}(A)$ with A a local complete noetherian ring and that $s_0 \in S$ is the closed point of S, $s_1 \in S$ is any point of S. Then one has the following canonical diagram of profinite groups, which commutes up to inner automorphisms:

$$
\begin{tikzcd}
(3) & 1 \arrow[r] & \pi_1(\overline{X}_0, \overline{x}_0) \arrow[r, "i_0"] & \pi_1(X, x_{(0)}) \arrow[r, "p_0"] & \pi_1(S, \overline{s}_0) \arrow[r] & 1 \\
(4) & & \pi_1(\overline{X}_1, \overline{x}_1) \arrow[r, "i_1"] \arrow[u, dotted, "\exists !\, sp"'] & \pi_1(X, x_{(1)}) \arrow[r, "p_1"] \arrow[u, "\alpha_X"'] & \pi_1(S, \overline{s}_1) \arrow[r] \arrow[u, "\alpha_S"'] & 1,
\end{tikzcd}
$$

where the vertical arrows $\alpha_X : \pi_1(X, x_{(1)}) \tilde{\to} \pi_1(X, x_{(0)})$ and $\alpha_S : \pi_1(S, \overline{s}_{(1)}) \tilde{\to} \pi_1(S, s_{(0)})$ are the canonical (up to inner automorphisms) isomorphisms of theorem 1.3.

Now, since $p_0 \circ \alpha_X \circ i_1 " = " \alpha_S \circ p_1 \circ i_1 \overset{(*)}{=} 0$ (here " = " means equal up to inner automorphisms and equality $(*)$ comes from theorem 3.11), one has $\mathrm{Im}(\alpha_X \circ i_1) \subset \mathrm{Ker}(p_0) = \mathrm{Im}(i_0)$ (by theorem 3.10) and, hence, there exists a profinite group morphism:

$$
sp : \pi_1(\overline{X}_1, \overline{x}_1) \to \pi_1(\overline{X}_0, \overline{x}_0),
$$

unique up to inner automorphisms and such that $\alpha_X \circ p_1 " = " i_0 \circ sp$.

If, furthermore, $\mathrm{Im}(i_1) = \mathrm{Ker}(p_1)$, a straightforward diagram chasing shows that $sp : \pi_1(\overline{X}_1, \overline{x}_1) \twoheadrightarrow \pi_1(\overline{X}_0, \overline{x}_0)$ is an epimorphism.

Construction of the specialization morphism-2. We come back to the case where S is any locally noetherian sheme and s_0, $s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$. One then has the following canonical commutative diagram:

$$
\begin{tikzcd}
\overline{k(s_1)} \arrow[r] & k(s_1) \arrow[r, "s_1"] & S & k(s_0) \arrow[l, "s_0"'] & \overline{k(s_0)} \arrow[l] \\
\overline{k(\hat{s}_1)} \arrow[r] \arrow[u] & k(\hat{s}_1) \arrow[r, "\hat{s}_1"'] \arrow[u] & \mathrm{spec}(\widehat{\mathcal{O}}_{S,s_0}) \arrow[u] & k(\hat{s}_0) \arrow[l, "\hat{s}_0"] \arrow[u, equal] & \overline{k(\hat{s}_0)}. \arrow[l] \arrow[u, equal]
\end{tikzcd}
$$

From the preceding §, one has a canonical specialization morphism:

$$
sp : \pi_1(\overline{X}_1 \times_{\overline{k(s_1)}} \overline{k(\hat{s}_1)}, \bar{\hat{x}}_1) \to \pi_1(\overline{X}_0, \overline{x}_0)
$$

and , from corollary 3.12, the canonical morphism $\pi_1(\overline{X}_1 \times_{\overline{k(s_1)}} \overline{k(\hat{s}_1)}\, \bar{\hat{x}}_1) \tilde{\to} \pi_1(\overline{X}_1, \overline{x}_1)$ is an isomorphism.

The two next sections will be devoted to (sketch of) proofs of theorem 3.10, theorem 3.11 and corollary 3.12.

3.2.2 First homotopy sequence

The proof resorts to deep results from [EGA3]; we will only sketch it but give references for the missing details.

Step 1: *Assuming that A is a local artinian ring, the conclusions of theorem 3.10 hold.*

Recall that, in an Artin ring, any prime ideal is maximal hence the nilradical and the Jacobson radical coincide. In particular, if A is local, the nilpotent elements of A are precisely those of its maximal ideal. From theorem 3.2, one may thus assume that $A = k(s_0)$ and, then, the conclusion $\pi_1(S, \overline{s}_0) \simeq \Gamma_{k(s_0)}$ is straightforward. Let $k(s_0)^i$ denote the inseparable closure of $k(s_0)$ in $\overline{k}(s_0)$ and $X_0^i := X \times_S k(s_0)^i$. Then the cartesian diagram:

$$\begin{tikzcd} \overline{X}_0 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[dr, phantom, "\square"] & S \\ \overline{X}_0 \arrow[r] & X_0^i \arrow[r] \arrow[u] & \mathrm{Spec}(k(s_0)^i) \arrow[u] \end{tikzcd} \tag{5}$$

induces a commutative diagram of profinite group morphisms:

$$\begin{tikzcd} \pi_1(\overline{X}_0, \overline{x}_0) \arrow[r] & \pi_1(X, x_{(0)}) \arrow[r] & \pi_1(S, \overline{s}_0) \\ \pi_1(\overline{X}_0, \overline{x}_0) \arrow[r] \arrow[u] & \pi_1(X_0^i, x_{(0)}^i) \arrow[r] \arrow[u] & \pi_1(\mathrm{Spec}(k(s_0)^i), \overline{s}_0^i) \arrow[u] \end{tikzcd} \tag{6}$$

Now, since each of the vertical arrows in (5) is faithfully flat, quasi-compact and radiciel, it follows from corollary 3.4 that the vertical arrows in (6) are profinite group isomorphisms. Hence it is enough to prove that the bottom line of (6) is exact that is one may assume that $k(s_0)$ is perfect.

But, then, $k(s_0)$ can be written as the inductive limit of its finite Galois subextensions $\{k(s_0) \hookrightarrow k_i\}_{i \in I}$ hence, writing $X_i := X \times_S k_i$ and x_i for the image of $\overline{x}_0$ in X_i, the base change functor $\varinjlim R^{et}_{X_i} \approx R^{et}_{\overline{X}}$ is an equivalence of categories hence induces a profinite group isomorphism

$$\pi_1(\overline{X}_0, \overline{x}_0) \tilde{\to} \varprojlim \pi_1(X_i, x_i).$$

But, for each $i \in I$, $X_i \to X \in R^{et}_X$ is Galois with group $\mathrm{Gal}(k_i | k(s_0))$ so, from proposition 1.19 one has a short exact sequence of profinite groups:

$$1 \to \pi_1(X_i, x_i) \to \pi_1(X, x_{(0)}) \to \mathrm{Gal}(k_i | k(s_0)) \to 1.$$

Using that the projective limit functor is exact in the category of profinite groups, we thus get the expected short exact sequence of profinite groups:

$$1 \to \varprojlim \pi_1(X_i, x_i) \to \pi_1(X, x_{(0)}) \to \Gamma_{k(s_0)} \to 1.$$

Remark 3.13 The above (with $A = k$ a field) shows that the statement of proposition 2.13 remains true without the assumption that k is perfect.

Step 2: *The closed immersion $i_{X_0} : X_0 \hookrightarrow X$ induces an equivalence of categories $R^{et}_X \approx R^{et}_{X_0}$. In particular, one has a canonical profinite groups isomorphism $\pi_1(X_0, x_0) \tilde{\to} \pi_1(X, x_{(0)})$.*

One has to prove that:
(i) For any $p : Y \to X$, $p' : Y' \to X \in R^{et}_X$ the canonical map

$$\mathrm{Hom}_{R^{et}_X}(p, p') \to \mathrm{Hom}_{R^{et}_{X_0}}(p \times_X X_0, p' \times_X X_0)$$

is an isomorphism and
(ii) For any $p_0 : Y_0 \to X_0 \in R^{et}_{X_0}$ there exists $p : Y \to X \in R^{et}_X$ such that $p \times_X X_0 \tilde{\to} p_0$ in $R^{et}_{X_0}$.

The proof of (i) and (ii) is based on Grothendieck's Comparison and Existence theorems in algebraico-formal geometry. We first state simplified versions of these theorems.

Let S be a noetherian scheme and $p : X \to S$ be a proper morphism. Let $\mathcal{I} \subset \mathcal{O}_S$ be a coherent sheaf of ideals. Then the descending chains $\cdots \subset \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \cdots \subset \mathcal{I}$ corresponds to a chain of closed subschemes $S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow \cdots \hookrightarrow S$. We will use the notation in the diagram below:

$$\begin{tikzcd} S & S_n \arrow[l, hook'] & \cdots \arrow[l, hook'] & S_1 \arrow[l, hook'] & S_0 \arrow[l, hook'] \\ X \arrow[u, "p"] \arrow[ur, phantom, "\square"] & X_n \arrow[l, hook'] \arrow[u, "p_n"] \arrow[ur, phantom, "\square"] & \cdots \arrow[l, hook'] & X_1 \arrow[l, hook'] \arrow[u, "p_1"] \arrow[ur, phantom, "\square"] & X_0. \arrow[l, hook'] \arrow[u, "p_0"] \end{tikzcd}$$

For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, set $\mathcal{F}_n := p_n^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$, $n \geq 0$. Then $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module and the canonical $\mathcal{O}_X$-module morphim $\mathcal{F} \to \mathcal{F}_n$ induces $\mathcal{O}_S$-module morphisms $\mathrm{R}^q p_*\mathcal{F} \to \mathrm{R}^q p_*\mathcal{F}_n$, $q \geq 0$ hence $\mathcal{O}_{S_n}$-module morphisms:

$$(\mathrm{R}^q p_*\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{X_n} \to \mathrm{R}^q p_*\mathcal{F}_n,\ q \geq 0$$

and, taking projective limit, canonical morphisms:

$$\varprojlim (\mathrm{R}^q p_*\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{X_n} \to \varprojlim \mathrm{R}^q p_*\mathcal{F}_n,\ q \geq 0.$$

When $S = \mathrm{spec}(A)$ is affine and $I \subset A$ is the ideal corresponding to $\mathcal{I} \subset \mathcal{O}_S$, the above isomorphism become:

$$\mathrm{H}^q(X, \mathcal{F}) \otimes_A \widehat{A} \tilde{\to} \varprojlim \mathrm{H}^q(X_n, \mathcal{F}_n),\ q \geq 0,$$

where $\widehat{A}$ denotes the completion of A with respect to the I-adic topology.

Theorem 3.14 (Comparison theorem [EGA3, (4.1.5)]) *The canonical morphisms:*

$$\varprojlim (\mathrm{R}^q p_*\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{X_n} \tilde{\to} \varprojlim \mathrm{R}^q p_*\mathcal{F}_n,\ q \geq 0$$

are isomorphisms.

Theorem 3.15 (Existence theorem [EGA3, (5.1.4)]) *Assume, furthermore that $S = \mathrm{spec}(A)$ is affine and that A is complete with respect to the I-adic topology. Let $\mathcal{F}_n$, $n \geq 0$ be coherent $\mathcal{O}_{X_n}$-modules such that $\mathcal{F}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n} \tilde{\to} \mathcal{F}_n$, $n \geq 0$. Then there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \mathcal{F}_n$, $n \geq 0$.*

For any $p : Y \to X \in R^{et}_X$, recall that $\mathcal{A}(p) := f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-algebra of finite rank and that the functor

$$\begin{array}{rlcl} \mathcal{A}: & R^{et}_X & \to & FLFA_{\mathcal{O}_X} \\ & p : Y \to X & \to & \mathcal{A}(p) \end{array}$$

is fully faithful.

-Proof of (i): Let $\mathcal{M}$ denote the maximal ideal of A and, for any $n \geq 0$, write $A_n := A/\mathcal{M}^{n+1}$. Then one has canonical functorial isomorphisms:

$$\begin{array}{rcl} \mathrm{Hom}_{R_X^{et}}(p,p') & \tilde{\to} & \mathrm{H}^0(X, \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_X}}(\mathcal{A}(p'),\mathcal{A}(p))) \\ & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_X}}(\mathcal{A}(p'),\mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}), \end{array}$$

where the first isomorphism comes from the fact that $\mathcal{A}$ is fully faithful and the second isomorphism is just the comparison theorem applied to $q=0$, $\mathcal{F} = \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_X}}(\mathcal{A}(p'),\mathcal{A}(p))$ and $I = \mathcal{M}$, observing that, since A is complete with respect to the $\mathcal{M}$-adic topology, $A = \widehat{A}$.

Furthermore, as $\mathcal{A}(p)$, $\mathcal{A}(p')$ are locally free $\mathcal{O}_X$-module, one has canonical isomorphisms:

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X-Mod}(\mathcal{A}(p'),\mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \underline{\mathrm{Hom}}_{\mathcal{O}_{X_n}-Mod}(\mathcal{A}(p'_n),\mathcal{A}(p_n))$$

But these preserve the structure of $\mathcal{O}_X$-algebra morphisms hence one also gets, by restriction:

$$\underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_X}}(\mathcal{A}(p'),\mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_{X_n}}}(\mathcal{A}(p'_n),\mathcal{A}(p_n)).$$

Whence,

$$\begin{array}{rcl} \mathrm{Hom}_{R_X^{et}}(p,p') & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_X}}(\mathcal{A}(p'),\mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}) \\ & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFA_{\mathcal{O}_{X_n}}}(\mathcal{A}(p'_n),\mathcal{A}(p_n))) \\ & \tilde{\to} & \varprojlim \mathrm{Hom}_{R_{X_n}^{et}}(p_n,p'_n) \\ & \tilde{\to} & \varprojlim \mathrm{Hom}_{R_{X_0}^{et}}(p_0,p'_0), \end{array}$$

where the last isomorphism comes from the fact $\mathrm{Hom}_{R_{X_n}^{et}}(p_n,p'_n) \tilde{\to} \mathrm{Hom}_{R_{X_0}^{et}}(p_0,p'_0)$, $n \geq 0$ by theorem 3.2.

-Proof of (ii): By theorem 3.2, there exists $p_n : Y_n \to X_n \in R_{X_n}^{et}$, $n \geq 0$ such that $p_n \tilde{\to} p_{n+1} \times_{X_{n+1}} X_n$, or, equivalently, $\mathcal{A}(p_{n+1}) \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n} \tilde{\to} \mathcal{A}(p_n)$, $n \geq 0$. So, by the Existence theorem, there exists $\mathcal{A} \in FLFA_{\mathcal{O}_X}$ such that $\mathcal{A} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_X \tilde{\to} \mathcal{A}(p_n)$, $n \geq 0$ hence, setting $p : Y = \mathrm{spec}\ (\mathcal{A}) \to X$ one has $p \times_X X_0 \tilde{\to} p_0$.

Claim: *One has $p : Y = \mathrm{spec}\ (\mathcal{A}) \to X \in R_X^{et}$.*

See [Mur67, p. 159-161].

Step 3: From step 1 applied to $A = k(s_0)$, $X = X_0$, one gets the short exact sequence of profinite groups:

$$1 \to \pi_1(\overline{X}_0, \overline{x_0}) \to \pi_1(X_0, x_0) \to \Gamma_{k(s_0)} \to 1.$$

Now, from step 2 one has the canonical profinite group isomorphisms $\pi_1(X,x) \tilde{\to} \pi_1(x_0,x_0)$ and (for $X = S$) $\pi_1(S,s_0) \tilde{\to} \Gamma_{k(s_0)}$, which yields the required short exact sequence.

Eventually, for the last assertion of theorem 3.10, just observe that, as above, one can assume that $A = k(s_0)$ thus, if $x \in X(k(s_0))$, it produces a section $x : S \to X$ of $f : X \to S$ such that $x \circ s_0 = x$ thus a section $\Gamma_{k(s_0)} \to \pi_1(X,x)$ of (3). $\square$

3.2.3 Second homotopy sequence

Technical preliminaries. Let k be a field. A k-scheme X is *separable over k* if, for any field extension K/k, $X\times_k K$ is reduced. This is equivalent to requiring that X be reduced and that, for any generic point $\eta\in X$, the extension $k(\eta)/k$ be separable (thus, if k is perfect, this is equivalent to requiring that X be reduced). Let S be a scheme. A S-scheme $X\rightarrow S$ is *separable over S* if $X\rightarrow S$ is flat over S and for any $s\in S$, $X\times_{S,s} k(s)$ is separable over $k(s)$.

Note that:

- Any base change of a separable morphism is separable.
- If $X\rightarrow S$ is separable over S and $X'\rightarrow X$ is etale over X then $X'\rightarrow S$ is separable over S.

Theorem 3.16 (Stein factorization of a proper morphism) *Let S be a locally noetherian scheme and $f:X\rightarrow S$ be a morphism. Then the coherent $\mathcal{O}_S$-algebra $f_*\mathcal{O}_X$ defines a S-scheme $p:S'=$* $\mathrm{spec}(f_*\mathcal{O}_X)\rightarrow S$ *and $f:X\rightarrow S$ fators canonically as:*

$$\begin{tikzcd} S & S' \arrow[l, "p"'] \\ X. \arrow[u, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Furthermore,

- If $f:X\rightarrow S$ is proper then $p:S'=\mathrm{spec}(f_*\mathcal{O}_X)\rightarrow S$ is finite and $f':X\rightarrow S'$ is proper and geometrically connected;

- If $f:X\rightarrow S$ is proper and separable then $p:S'=\mathrm{spec}(f_*\mathcal{O}_X)\rightarrow S\in R_S^{et}$.

Corollary 3.17 *Let S be a locally noetherian scheme and $f:X\rightarrow S$ be a proper morphism with Stein factorization $X\xrightarrow{f'}S'\xrightarrow{p}S$. Then,*

- If $f:X\rightarrow S$ is proper then for any $s\in S$, the connected components of $X\times_{f,S,s}k(s)$ are one-to-one with the finite set of points above s and the connected components of $X\times_{f,S,\overline{s}}\overline{k(s)}$ are one-to-one with the finite set of geometric points above s. In particular, if $f_*\mathcal{O}_X=\mathcal{O}_S$ then X is connected.

- If $f:X\rightarrow S$ is proper and separable then $f_*\mathcal{O}_X=\mathcal{O}_S$ if and only if $f:X\rightarrow S$ is geometrically connected.

Proof. Exactness on the right.

Lemma 3.18 (Flat base change) *Let $f:X\rightarrow S$ be a proper morphism and let $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for any cartesian diagram:*

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "\pi"'] & S \end{tikzcd}$$

with $\pi:S'\rightarrow S$ flat over S, one has canonical isomorphisms:

$$c_n:\pi^*R^qf_*(\mathcal{F})\tilde{\rightarrow}R^qf'_*({\pi'}^*\mathcal{F}),\ q\geq 0$$

In particular, for any $\pi : S' \to S \in R_S^{et}$ with S' connected, one has:

$$f'_*(\mathcal{O}_{X'}) = f'_*(\pi_X^*\mathcal{O}_X) \overset{(1)}{=} \pi^* f_*\mathcal{O}_X \overset{(2)}{=} \pi^*\mathcal{O}_S = \mathcal{O}_{S'},$$

where (1) results from the flatness of $\pi : S' \to S$ and lemma 3.18 whereas (2) results from the assumption $f_*\mathcal{O}_X = \mathcal{O}_S$. Now, as $f' : X' \to S'$ is proper, it follows from corollary 3.17 that X' is connected.

Exactness in the middle.

- $\underline{\ker(p_1) \supset \operatorname{im}(i_1)}$: Let $\pi : S' \to S \in R_S^{et}$ and consider the following notation:

$$\begin{tikzcd}
\overline{S}'_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S'_{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S' \arrow[d, leftrightarrow] \\
\overline{X}_1 \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d, phantom, "\square"] & S \\
 & \overline{k(s_1)} \arrow[ur] &
\end{tikzcd}.$$

Then

$$\begin{aligned}
\overline{S}'_1 \quad = \overline{X}_1 \times_S S' \quad &= (X \times_{S,\overline{s}_1} \overline{k(s_1)}) \times_S S' \\
&= X \times_S (\overline{k(s_1)}) \times_{\overline{s}_1,S} S') \\
&= X \times_S \coprod_{finite} \overline{k(s_1)}) \\
&= \coprod_{finite} \overline{X}_1.
\end{aligned}$$

- $\underline{\ker(p_1) \subset \operatorname{im}(i_1)}$: Let $\pi : X' \to X \in R_X^{et}$ with X' connected. Consider the following notation:

$$\begin{tikzcd}
\overline{X}'_1 \arrow[r, "\pi"] \arrow[dr, phantom, "\square"] & X \arrow[r, "f"] \arrow[dr, phantom, "\square"] & S \\
\overline{X}'_1 \arrow[r, "\overline{\pi}_1"'] \arrow[u] & \overline{X}_1 \arrow[r] \arrow[u] & \overline{k(s_1)} \arrow[u]
\end{tikzcd}$$

and assume that there exists a section $\sigma : \overline{X}_1 \to \overline{X}'_1$ of $\overline{\pi}_1 : \overline{X}'_1 \to \overline{X}_1$.

Since $\pi : X' \to X$ is etale and $f : X \to S$ is proper and separable, $g := f \circ \pi : X' \to S$ is also proper and separable. Consider its Stein factorization $X' \xrightarrow{g'} S' \xrightarrow{p} S$. From theorem 3.16, $p : S' \to S$ is etale over S. Consider now the following commutative diagram:

$$\begin{tikzcd}
 & X' \arrow[dl, "\pi"'] \arrow[d, dotted, "\alpha"] \arrow[dd, bend left, "g'"] \\
X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X'' \arrow[l, "pr_X"] \arrow[d, "f'"] \\
S & S' \arrow[l, "p"]
\end{tikzcd}. \tag{7}$$

Then $\alpha : X \tilde{\to} X''$ *is an isomorphism.* Indeed,

(i) Since any base change of an etale morphism is again etale, $pr_X : X'' \to X$ is etale. Also, by hypothesis, $\pi_: X' \to X$ is etale hence $\alpha : X \to X''$ *is etale.*

(ii) Since X' is connected and $g' : X' \to S'$ is surjective, S' is also connected. But then, it follows from the surjectivity of $p_1 : \pi_1(X, x_{(1)}) \twoheadrightarrow \pi_1(S, \overline{s}_1)$ that X'' *is also connected.*

(iii) We now base change (7) *via* $\overline{s}_1 : \overline{k(s_1)} \to S$.

$$\begin{tikzcd}
 & \overline{X}'_1 \arrow[dl, "\overline{\pi}_1"'] \arrow[d, "\overline{\alpha}_1"] \arrow[dd, bend left, "\overline{g'}_1"] \\
\overline{X}_1 \arrow[ur, bend left, "\sigma"] \arrow[d, "\overline{f}_1"'] & \overline{X}''_1 \arrow[l, "\overline{pr}_{X\,1}"] \arrow[d, "\overline{f'}_1"] \arrow[dl, phantom, "\square"] \\
\overline{k(s_1)} & \overline{S}'_1 \arrow[l, "\overline{p}_1"]
\end{tikzcd} \quad . \tag{8}$$

Since $p : S' \to S$ is an etale cover, $\overline{S'}_1 = \coprod_{\overline{S}'_s} \mathrm{spec}(\overline{k(s_1)})$ hence $\overline{X}''_1 = \coprod_{\overline{S}'_s} \overline{X}_1$. Since $\overline{X}_1$ is connected and $\sigma : \overline{X}_1 \to \overline{X}'_1$ is etale hence maps connected components to connected components, $\sigma(\overline{X}_1) = Y'$ is a connected component of $\overline{X}'_1$. Again, since $\overline{\alpha}_1 : \overline{X}'_1 \to \overline{X}''_1$ is etale, $\overline{\alpha}_1(Y')$ is one of the connected component $Y'' \simeq \overline{X}_1$ of $\overline{X}'_1$. Eventually, since $\overline{pr}_{X\,1}|_{Y''} : Y'' \tilde{\to} \overline{X}_1$ is an isomorphism, one gets that $\overline{\alpha}_1|_{Y'} : Y' \tilde{\to} Y''$ is an isomorphism with inverse $\sigma \circ \overline{pr}_{X\,1}|_{Y''} : Y''' \tilde{\to} Y'$. Now, $|\pi_0(\overline{X}'_1)| = |\pi_0(\overline{X}''_1)| = |\overline{S}'_1|$ and $\overline{\alpha}_1 : \overline{X}'_1 \to \overline{X}_1$ is surjective (since it is both closed and open and ($\overline{X}''_1$ is connected) and etale so it induces a bijection $\pi_0(\overline{X}'_1) \tilde{\to} \pi_0(\overline{X}''_1)$ and, in particular, for any $y'' \in Y''$ $|\overline{\alpha}_s^{-1}(y'')| = 1$. So $\alpha : X' \to X''$ *has rank* 1 *at* y''.

Combining (i), (ii) and (iii), one gets that $\alpha : X' \tilde{\to} X''$ is an isomorphism. $\square$

As a result, $\alpha^{-1} \in \mathrm{Hom}_{R_X^{et}}(pr_X, \pi)$ as required.

Remark 3.19 The assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ can be omitted and the conclusion of theorem **??** then becomes that the following canonical exact sequence of profinite groups is exact:

$$\pi_1(\overline{X}_1, \overline{x}_1) \xrightarrow{i_1} \pi_1(X, x_{(1)}) \xrightarrow{p_1} \pi_1(S, \overline{s}_1) \to \pi_0(\overline{X}_1) \to \pi_0(X) \to \pi_0(S) \to 1$$

Proof of corollary 3.12.

Lemma 3.20 $X \times_k Y$ *is connected.*

Proof. Since the question is purely topological, one may assume that $X = X^{red}$ thus that $X \to k$ is separable. As $p_2 : X \times_k Y \to Y$ is proper (since $X \to k$ is) and surjective and as Y is connected, it is enough to prove that the fibres of $p_2 : X \times_k Y \to Y$ are connected. For this, it is enough to show that for any field extension K/k, $X \times_k K$ is connected. Since X is reduced, connected and k is algebraically closed, one has $\mathrm{H}^0(X, \mathcal{O}_X) = k$ (Stein factorization) hence, $\mathrm{H}^0(X \times_k K, \mathcal{O}_{X \times_k K}) = \mathrm{H}^0(X, \mathcal{O}_X) \otimes_k K = K$ (flat base change) but, as $X \times_k K$ is reduced, this implies that $X \times_k K$ is connected (Stein factorization). $\square$

From theorem 3.2, the closed immersion $X^{red} \hookrightarrow X$ induces an equivalence of categories $R_X^{et} \approx R_{X^{red}}^{et}$ so one can assume that $X = X^{red}$ thus that $X \to k$ is separable.

Since $X \to k$ is proper and separable, by base change, $p_2 : X \times_k Y \to Y$ is also proper and separable and, as already noticed, geometrically connected. It then follows from the cartesian diagram:

$$\begin{tikzcd}
\Omega \arrow[r, "{(a,b)}"] & X_\Omega \times_\Omega Y \arrow[r, "p_2"] & Y \\
\Omega \arrow[u] \arrow[r, "a"'] \arrow[ur, phantom, "\square"] & X_\Omega \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \Omega \arrow[u, "b"']
\end{tikzcd}$$

and from theorem 3.11 that one has a canonical exact sequence of profinite groups:

$$\pi_1(X_\Omega, a) \to \pi_1(X_\Omega \times_\Omega Y, (a, b)) \to \pi_1(Y, b) \to 1.$$

Then, the composite $X_\Omega \hookrightarrow X_\Omega \times_\Omega Y \overset{p_1}{\to} X_\Omega$ is the identity hence $\pi_1(X_\Omega, a) \to \pi_1(X_\Omega \times_\Omega Y, (a, b))$ admits a continuous group theoretic section and, in particular, is injective. So, actually, one gets a split short exact sequence of profinite groups:

$$1 \to \pi_1(X_\Omega, a) \to \pi_1(X_\Omega \times_\Omega Y, (a, b)) \to \pi_1(Y, b) \to 1.$$

Thus the conclusion will follow from:

Lemma 3.21 *The canonical morphism $\pi_1(X_\Omega, a_\Omega)\tilde{\to}\pi_1(X, a)$ is an isomorphism.*

Proof. First, from lemma 3.20 for $Y = \mathrm{spec}(\Omega)$, X_Ω is connected. Similarly, for any $\pi : X' \to X \in R_X^{et}$ with X' connected, $X' \to k$ is proper (as the composite of a finite morphism with a proper morphism) hence, X'_Ω is connected as well, which shows that $\pi_1(X_\Omega, a) \twoheadrightarrow \pi_1(X, a)$ is an epimorphism. So, it only remains to prove that for any $\overline{\pi} : X'_\Omega \to X_\Omega \in R_{X_\Omega}^{et}$, there exists $\pi : X' \to X \in R_X^{et}$ such that $\pi \times_k \Omega \tilde{\to} \overline{\pi}$. First, one can always find a k-subalgebra $A \subset \Omega$ of finite type over k and $\pi_A : X'_A \to X_A \in R_{X_A}^{et}$ such that $\pi_A \times_A \Omega \tilde{\to} \overline{\pi}$. Set $Y := \mathrm{spec}(A)$ (hence $X_A = X \times_k Y$); since $Y \to k$ is of finite type and k is algebraically closed, one can always find $b_k \in Y$ such that $k(b_k) = k$. Also, since the fundamental group does not depend on the fibre functor, one can assume that $k(a) = k$. Then, from the above, one gets the canonical profinite group isomorphism:

$$\pi_1(X \times_k Y, (a, b_k))\tilde{\to}\pi_1(X, a) \times \pi_1(Y, b_k).$$

Let $U \subset \pi_1(X \times_k Y, (a, b_k))$ be the open subgroup corresponding to $\pi_A : X'_A \to X \times_k Y \in R_{X \times_k Y}^{et}$. Then consider two open normal subgroups $U_X \lhd \pi_1(X, a)$, $U_Y \lhd \pi_1(Y, b_k)$ such that $U_X \times U_Y \subset U$. $U_X \lhd \pi_1(X, a)$, $U_Y \lhd \pi_1(Y, b_k)$ correspond to Galois covers $\hat{\pi}_1 : \hat{X} \to X \in R_X^{et}$ and $\hat{\pi}_2 : \hat{Y} \to Y \in R_Y^{et}$ respectively and $\pi_A : X'_A \to X \times_k Y \in R_{X \times_k Y}^{et}$ is a quotient of $\hat{\pi}_1 \times_k \hat{\pi}_2 : \hat{X} \times_k \hat{Y} \to X \times_k Y$. Consider now the following cartesian diagram:

$$\begin{tikzcd}
 & & \hat{X} \times_k \hat{Y} \arrow[dll] \arrow[dl, dotted] \arrow[ddl] \\
X'_A \arrow[d] \arrow[dr, phantom, "\square"] & X''_A \arrow[l] \arrow[d] & \\
X \times_k Y & X \times_k \hat{Y} \arrow[l] &
\end{tikzcd}$$

and let $y \in Y$ be the generic point of Y. Since $k(y) \subset \Omega$ and Ω is algebraically closed, one may assume that for any point $\hat{y} \in \hat{Y}$ above $y \in Y$, $k(\hat{y}) \subset \Omega$ and that one has the cartesian diagram:

$$\begin{tikzcd} X''_A \arrow[d] & & X'_\Omega \arrow[ll] \arrow[d] \\ X \times_k \hat{Y} \arrow[rr, phantom, "\square"] & (X \times_k \hat{Y}) \times_{\hat{y},\kappa(\hat{y})} \Omega \simeq X \times_k \Omega. \arrow[l] & \end{tikzcd}$$

Thus, as X'_Ω is connected, so is X''_A, from which it follows in particular that $\hat{X} \times_k \hat{Y} \to X''_A$ is surjective and that $X''_A \to X \times_k \hat{Y}$ corresponds to an open subgroup $V \subset \pi_1(X \times_k \hat{Y}) = \pi_1(X) \times U_Y$ containing $\pi_1(\hat{X} \times_k \hat{Y}) = U_X \times U_Y$. Hence $X''_A \to X \times_k \hat{Y}$ is of the form $X' \times_k \hat{Y} \to X \times_k \hat{Y}$ for some $\pi : X' \to X \in R^{et}_X$. Now, on the one hand $(X''_A)_{\hat{y}} \tilde{\to} X' \times_k k(\hat{y})$ and, on the other hand, $X'_\Omega \tilde{\to} X'_A \times_Y \Omega \tilde{\to} (X'_A)_y \times_{k(y)} \Omega \tilde{\to} (X''_A)_{\hat{y}} \times_{k(\hat{y})} \Omega \simeq X' \times_k \Omega$. $\square$

3.2.4 Proper schemes over algebraically closed fields

Let k be an algebraically closed field and $X \to k$ a proper morphism with X connected. Then:

Theorem 3.22 *For any geometric point $x \in X$, $\pi_1(X, x)$ is finitely generated.*

Proof. The proof is by induction on $\dim(X) = d$.

1. Reduction to the case where X is connected, normal and projective over k. The main argument is Chow's lemma [EGA2, Cor. 5.6.2], which state that for any scheme X proper over a noetherian scheme S there exists a scheme $\tilde{X}S$ projective over S and a surjective birational morphism $\tilde{X} \to X$. Write $\tilde{X}^{red} \hookrightarrow \tilde{X}$ for the underlying reduced closed subscheme and $\overline{\tilde{X}}^{red} \to \tilde{X}^{red}$ for its normalization. The resulting morphism $\overline{\tilde{X}}^{red} \to X$ is then surjective and proper as the composite of three surjective and proper morphisms. (Indeed, the surjectivity is straigtforward. As for the properness: since both X and $\tilde{X}$ are proper over k, so is the morphism $\tilde{X} \to X$, $\tilde{X}^{red} \hookrightarrow \tilde{X}$ is a closed immersion, hence is proper and since $\tilde{X}^{red}$ is of finite type over k, $\overline{\tilde{X}}^{red} \to \tilde{X}^{red}$ is finite hence proper). In our situation, all the schemes have finitely many connected components so, by theorem 3.5 and corollary 3.7 it only remains to prove that $\pi_1(\overline{\tilde{X}}^{red}, x)$ is topologically of finite type.

   So, now, assume that X is connected, normal and projective over k.

2. $d = 0,\ 1$. If $d = 0$, there is nothing to prove. If $d = 1$, let Q denote the prime field of k. Since $X \to k$ is of finite type, there exists a subextension $Q \hookrightarrow k_0$ of $Q \hookrightarrow k$ with $\mathrm{transcdeg}(k_0|Q) < +\infty$ and a k_0-curve X_0 such that

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_0 \arrow[d] \\ k \arrow[r] & k_0. \end{tikzcd}$$

-char$(Q) = 0$. As transcdeg$(k_0|Q) < +\infty$, one can find a field embedding $k_0 \hookrightarrow \mathbb{C}$ hence, from lemma 3.21,

$$\pi_1(X) \simeq \pi_1(X_0 \times_{k_0} k) \simeq \pi_1(X_0 \times_{k_0} \overline{k}_0) \simeq \pi_1(X_0 \times_{k_0} \mathbb{C}).$$

So, one can assume that $k = \mathbb{C}$. It then follows from example 2.22 that one gets a profinite group isomorphism:

$$\pi_1(X)\tilde{\to}\widehat{\Gamma}_{g,0},$$

where g denotes the genus of X.

-char$(Q) = p > 0$. Let $W(k)$ be the ring of Witt vectors over k; it is a complete discrete valuation ring with residue field k and fraction field K of characteristic 0. Then there exists a smooth projective $W(k)$-scheme $W(X) \to W(k)$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W(X) \arrow[d] \\ k \arrow[r] & W(k). \end{tikzcd}$$

So, if s_1 and s_0 denote the generic and closed points of spec$(W(k))$ respectively, one gets with the notation of theorem 3.9, a profinite group epimorphism:

$$sp : \pi_1(\overline{W(X)}_1) \twoheadrightarrow \pi_1(\overline{W(X)}_0 = X).$$

But $\overline{W(X)}_1 \to \overline{K}$ is also a $\overline{K}$-curve of genus g, hence one has constructed a profinite group epimorphism:

$$sp : \widehat{\Gamma}_{g,0} \twoheadrightarrow \pi_1(\overline{W(X)}_0 = X).$$

This proves the $d = 1$ case.

3. $d =\geq 2$. Let $X \hookrightarrow \mathbb{P}^n_k$ be a closed immersion and let $H \hookrightarrow \mathbb{P}^n_k$ be an hyperplane such that $X \not\subset H$ then the corresponding hyperplane section $X \cdot H$ (regarded as a scheme with the induced reduced scheme structure) has dimension $\leq d-1$ thus the conclusion will follow from:

   Lemma 3.23 *Let X be scheme proper over k, irreducible and normal and let $f : X \to \mathbb{P}^n_k$ be a k-morphism such that $\dim(g(X)) \geq 2$. Then, for any hyperplane $H \hookrightarrow \mathbb{P}^n_k$ the scheme $Y := X \times_{f,\mathbb{P}^n_k} H$ is connected and for any finite connected etale cover $X' \to X$, the induced finite etale cover $Y' := X' \times_X Y \to Y$ is again connected. In other words, the canonical profinite group morphism $\pi_1(Y, y) \to \pi_1(X, x)$ is an epimorphism.*

   Proof of the lemma. Since X is normal, X' is normal as well hence, being connected, it is also irreducible. Thus, if $\mathcal{H}$ is the generic hyperplane of $\mathbb{P}^n_k$ (defined over $K = k(T_0, \dots, T_n)$) then it follows from Bertini theorem that $X'_K \times_{\mathbb{P}^n_K} \mathcal{H}$ is universally irreducible hence, universally connected over K but then, it follows from Zariski connexion theorem that for any hyperplane $H \hookrightarrow \mathbb{P}^n_k$ (defined over any extension $k(H)$ of k) that $X'_{k(H)} \times_{\mathbb{P}^n_{k(H)}} \mathcal{H}$ is geometrically connected over $k(H)$. $\square$

Corollary 3.24 *For any finite group G there are only finitely many isomorphism classes of $p : X' \to X \in R^{et}_X$ Galois such that $\mathrm{Aut}_{R^{et}_X}(p) = G$.*

3.3 Purity and applications

3.3.1 The purity theorem and aplications

Theorem 3.25 (Zariski-Nagata purity theorem) *Let X, Y be integral schemes with X normal and Y regular and locally noetherian. Let $f : X \to Y$ be a quasi-finite dominant morphism and let $Z_f \subset X$ denote the closed subset of all $x \in X$ such that $f : X \to Y$ is not etale at x. Then, either $Z_f = X$ or Z_f is pure of codimension 1 (that is, for any generic point $\eta \in Z$, $\dim(\mathcal{O}_{X,\eta}) = 1$).*

Corollary 3.26 *Let X be a connected, regular, locally noetherian scheme and let $U \subset X$ be an open subset such that $X \setminus U$ has codimension ≥ 2 in X. Then the open immersion $U \hookrightarrow X$ induces an equivalence of categories $R_X^{et} \approx R_U^{et}$. In particular, for any geometric point $x \in U$, the canonical morphism*

$$\pi_1(U, x) \tilde{\to} \pi_1(X, x)$$

is an isomorphism.

Proof. As X is connected and regular (hence normal), X is irreducible. Since X is normal and $X \setminus U \subset X$ is a closed subset of codimension ≥ 2, the restriction functor $\mathrm{Mod}^{loclib}(\mathcal{O}_X) \to \mathrm{Mod}^{loclib}(\mathcal{O}_U)$ is fully faithfull hence, one only has to prove that for any finite etale cover $p_U : V \to U$ there exists a (necessarilly unique by the above) finite etale cover $p : Y \to X$ such that

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p_U"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

One may assume that V is connected hence, being normal (since U is), irreducible. So V is the normalization of U in $k(X) = k(U) \hookrightarrow k(V)$. Let $p : Y \to X$ be the normalization of X in $k(X) \hookrightarrow k(V)$. Then, on the one hand,

$$\begin{tikzcd} V \arrow[r] \arrow[d, "p_U"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] \\ U \arrow[r] & X \end{tikzcd}$$

and, on the other hand, since X is normal and $k(X) \hookrightarrow k(V)$ is a finite separable field extension, $p : Y \to X$ is finite, dominant and etale on $p^{-1}(U) = V = Y \setminus p^{-1}(X \setminus U)$. But $X \setminus U$ has codimension ≥ 2 in X hence, since $p : Y \to X$ is finite, $p^{-1}(X \setminus U)$ has codimension ≥ 2 in Y as well. Thus, from theorem 3.25 $p : Y \to X$ is etale. $\square$

Now, let X be a locally noetherian regular scheme and $f : X \rightsquigarrow Y$ be a rational map. Write $U_f \subset X$ for the maximal open subset on which $f : X \rightsquigarrow Y$ is defined and assume that $X \setminus U_f$ has codimension ≥ 2 in X. Then, one has the canonical functors:

$$R_Y^{et} \to R_{U_f}^{et} \approx R_X^{et}$$

and, correspondingly, for any geometric point $x \in U_f$, profinite group morphisms:

$$\pi_1(X, x) \tilde{\to} \pi_1(U_f, x) \to \pi_1(Y, f(x)).$$

Thus, if one consider the category $\mathcal{C}$ of all connected, locally noetherian, regular schemes pointed by geometric points in codimension 1 together with dominant rational maps defined on an open subscheme whose complement has codimension ≥ 2, one gets a well defined functor $\pi_1(-) : \mathcal{C} \to$ *ProGr*, $(X, x) \mapsto \pi_1(X, x)$. In particular,

Corollary 3.27 (Birational invariance of fundamental groups) *Let k be a field, X, Y two schemes proper over k and regular and $f : X \leftrightsquigarrow Y$ a birational k-map. Then, for any geometric point $x \in U_f$ one has the canonical profinite group isomorphisms*

$$\pi_1(X,x)\tilde{\to}\pi_1(U_f,x)\tilde{\to}\pi_1(U_{f^{-1}},f(x))\tilde{\to}\pi_1(Y,f(x)).$$

Proof. If k is a field, X a normal k-scheme and Y a scheme proper over k then any rational k-map $f : X \rightsquigarrow Y$ is defined over an open subset $U_f \subset X$ such that $X \setminus U_f$ has codimension ≥ 2. Thus the claim follows from corollary 3.26. □

Example 3.28 Let k be any field and consider the blowing-up $f : B_x \to \mathbb{P}^2_k$ of $\mathbb{P}^2_k$ at any point $x \in \mathbb{P}^2_k$. Then for any geometric point $b \in B_x$:

$$\pi_1(X,b)\tilde{\to}\pi_1(\mathbb{P}^2_k,f(b)).$$

However, B_x and $\mathbb{P}^2_k$ are not k-isomorphic (any two curves $\mathbb{P}^2_k$ intersects whereas the exceptional divisor E in B_x does not intersect the inverse images of the curves in $\mathbb{P}^2_k$ passing away from x). The above result is straightfroward in characteristic 0 since, combining Riemann Existence Theorem, specialization theory and the short exact sequence for geometrically connected schemes over fields, one gets : $\pi_1(X,x)\tilde{\to}\pi_1(\mathbb{P}^2_k,f(b))\tilde{\to}\Gamma_k$. But it is not in positive charactaristic and shows, in particular, the complexity of higher dimensional anabelian geometry.

3.3.2 Kernel of the specialization morphism

<u>**Ramification**</u> Recall that if $(\mathcal{O},\mathcal{M})$ is a discrete valuation ring with fraction field $K = \mathrm{Frac}(\mathcal{O})$ and residue field $k = \mathcal{O}/\mathcal{M}$ and if L/K is a finite Galois extension then the integral closure $\tilde{\mathcal{O}}^L$ of $\mathcal{O}$ in L is a free $\mathcal{O}$-module of rank $n = [L : K]$. For any maximal ideal $\mathcal{M}_L$ of $\tilde{\mathcal{O}}^L$ write $k_{\mathcal{M}_L} := \tilde{\mathcal{O}}^L/\mathcal{M}_L$ for the residue extension and:

$$D_{L/K}(\mathcal{M}_L) := \{\sigma \in \mathrm{Gal}(L|K) \mid \sigma(\mathcal{M}_L) = \mathcal{M}_L\}$$

for the *decomposition group of $\mathcal{M}_L$ in L/K.* Thus we have a canonical group epimorphism $D_{L/K}(\mathcal{M}_L) \twoheadrightarrow \mathrm{Gal}(k_{\mathcal{M}_L}|k)$ whose kernel is the *inertia group of $\mathcal{M}_L$ in L/K* and denoted by $I_{L/K}(\mathcal{M}_L)$. Since $\mathrm{Gal}(L|K)$ acts transitively on the maximal ideals of $\tilde{\mathcal{O}}^L$, the $D_{L/K}(\mathcal{M}_L)$ (resp. the $I_{L/K}(\mathcal{M}_L)$) form a whole conjugacy class $\mathfrak{D}_{L/K}(\mathcal{O})$ (resp. $\mathfrak{I}_{L/K}(\mathcal{O})$) of subgroups of $\mathrm{Gal}(L|K)$ so we will simply write $D_{L/K}(\mathcal{O})$ (resp. $I_{L/K}(\mathcal{O})$) for a representative of $\mathfrak{D}_{L/K}(\mathcal{O})$ (resp. $\mathfrak{I}_{L/K}(\mathcal{O})$). One says that L/K is *tamely ramified over $\mathcal{O}$* if $e_{L|K}(\mathcal{O}) := |I_{L/K}(\mathcal{O})|$ is prime to the characteristic of k and that L/K is *unramified over $\mathcal{O}$* if $e_{L|K}(\mathcal{O}) = 1$. We then have the following elementary properties:

Lemma 3.29 *Let π denote a uniformizing parameter of $\mathcal{M}$.*

1. *If L/K is tamely ramified over $\mathcal{O}$ then the canonical morphism*

$$\theta_0 : I_{L/K}(\mathcal{M}_L) \hookrightarrow (\tilde{\mathcal{O}}^L)^\star,\ \sigma \mapsto \frac{\sigma(\pi)}{\pi}\ \mathrm{mod}\mathcal{M}_L$$

   is a monomorphism and induces an isomorphism $\theta_0 : I_{L/K}(\mathcal{M})\tilde{\to}\mu_{e_{L|K}(\mathcal{O})}(K)$. In particular, $I_{L/K}(\mathcal{M}_L)$ is cyclic.

2. *(Transitivity) Let $K \subset L \subset M$ be finite field extensions with L/K and M/K Galois. Let $\mathcal{M}_M$ be a maximal ideal of $\tilde{\mathcal{O}}^M$ and $\mathcal{M}_L := \mathcal{M}_M \cap \tilde{\mathcal{O}}^L$. Then one has a commutative diagram with*

exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & I_{M/L}(\mathcal{M}_M) \arrow[r] \arrow[d] & I_{M/K}(\mathcal{M}_M) \arrow[r] \arrow[d] & I_{L/K}(\mathcal{M}_L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & D_{M/L}(\mathcal{M}_M) \arrow[r] \arrow[d] & D_{M/K}(\mathcal{M}_M) \arrow[r] \arrow[d] & D_{L/K}(\mathcal{M}_L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Gal}(M|L) \arrow[r] \arrow[d] & \mathrm{Gal}(M|K) \arrow[r] \arrow[d] & \mathrm{Gal}(L|K) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

3. *(Abhyankar's lemma) Let L/K and M/K be two finite Galois extensions tamely ramified over $\mathcal{O}$ and assume that $e_{L|K}(\mathcal{O})|e_{M|K}(\mathcal{O})$. Then, for any maximal ideal $\mathcal{M}_L$ of $\tilde{\mathcal{O}}^L$, $L.M$ is unramified over $\tilde{\mathcal{O}}^L_{\mathcal{M}_L}$.*

Example 3.30 Let $\pi \in \mathcal{M}$ be a uniformizing parameter and $n \geq 1$ an integer prime to the characteristic of k. Assume that K contains the nth roots of unity. Then $L := K[X]/ < X^n - \pi >$ is a finite Galois extension, tamely ramified over $\mathcal{O}$ and with Galois group $I_{L/K}(\mathcal{O}) \simeq \mathbb{Z}/n$.

Kernel of the specialization morphism We retain the notation of §3.2. Let S be a locally noetherian scheme and $f : X \to S$ a smooth, proper, geometrically connected morphism; our aim is to try and describe the kernel of the specialization epimorphism $\pi_1(\overline{X}_1, \overline{x}_1) \twoheadrightarrow \pi_1(\overline{X}_0, \overline{x}_0)$.

Theorem 3.31 *For any finite group G of order prime to the residue characteristic p of S at s_0 and for any profinite group epimorphism $\phi : \pi_1(\overline{X}_1, \overline{x}_1) \twoheadrightarrow G$ there exists a profinite group epimorphism $\phi_0 : \pi_1(\overline{X}_0, \overline{x}_0) \twoheadrightarrow G$ such that $\phi_0 \circ sp = \phi$. In particular, sp induces a profinite group isomorphism*

$$sp^{(p')} : \pi_1(\overline{X}_1, \overline{x}_1)^{(p')} \twoheadrightarrow \pi_1(\overline{X}_0, \overline{x}_0)^{(p')},$$

where $(-)^{(p')}$ denotes the prime-to-p profinite completion.

Proof. As one can always find a complete discrete valuation ring A with algebraically closed residue field and a morphism $\mathrm{spec}(A) \to S$ sending the generic point of $\mathrm{spec}(A)$ to s_1 and the closed point of $\mathrm{spec}(A)$ to s_0, one may assume without loss of generality that $S = \mathrm{spec}(A)$. Let K and k denote the fraction field and residue field of A respectively and let $K \hookrightarrow \overline{K}$ be an algebraic closure of K and $K \hookrightarrow K^s$ the separable closure of K in $K \hookrightarrow \overline{K}$. For any subring $A \subset B \subset \overline{K}$, we will write $X_B := X \times_A B$. For instance, $X = X_A$, $X_1 = X_K$, $\overline{X}_1 = X_{\overline{K}}$, $X_0 = \overline{X}_0 = X_{\overline{K}}$ *etc.*

Since the canonical morphism $\mathrm{spec}(\overline{K}) \to \mathrm{spec}(K^s)$ is faithfully flat, quasi-compact and radiciel, it follows from corollary 3.4 that the canonical profinite group morphism $\pi_1(X_{\overline{K}}, \overline{x}_1) \tilde{\to} \pi_1(X_{K^s}, x_1^s)$ is an isomorphism. Also, it follows from step 2 of the proof of theorem 3.10 that the canonical profinite group morphism $\pi_1(X_0, \overline{x}_0) \tilde{\to} \pi_1(X, \overline{x}_{(0)})$ is an isomorphism. Hence we are to determine the kernel of $\pi_1(X_{K^s}, \overline{x}_1^s) \twoheadrightarrow \pi_1(X, \overline{x}_{(0)})$ or, equivalently, to solve:

Problem: *Given $f^s : Y^s \to X_{K^s} \in R^{et}_{X_{K^s}}$ Galois with group G, when can we say that there exists $f : Y \to X \in R^{et}_X$ Galois with group G such that:*

$$\begin{tikzcd} Y^s \arrow[r] \arrow[d, "f^s"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] & ? \\ X_{K^s} \arrow[r] & X \end{tikzcd}$$

Since K^s is the inductive limit of the finite extensions of K contained in K^s, there exists a finite separable extension $K \hookrightarrow K_1$ and a finite etale Galois cover $f_1 : Y_1 \to X_{K_1} \in R^{et}_{X_{K_1}}$ such that:

$$\begin{tikzcd} Y^s \arrow[r] \arrow[d, "f^s"'] \arrow[dr, phantom, "\square"] & Y_1 \arrow[d, "f_1"] \\ X_{K^s} \arrow[r] & X_{K_1}. \end{tikzcd}$$

Thus the problem becomes:

Problem: *When does there exist a finite etale Galois cover $f : Y \to X \in R^{et}_X$ and a finite separable extension $K_1 \hookrightarrow K_2$ such that:*

$$\begin{tikzcd} Y_{1K_2} \arrow[r] \arrow[d, "f_{1K_2}"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] & ? \\ X_{K_2} \arrow[r] & X \end{tikzcd}$$

But, given any finite separable extension L/K the integral closure $\tilde{A}^L$ of A in L/K is again a complete discete valuation ring with residue field k. Hence, considering the cartesian square:

$$\begin{tikzcd} X_{\tilde{A}^L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{spec}(\tilde{A}^L) \arrow[r] & \operatorname{spec}(A) \end{tikzcd}$$

Writing $(X_{\tilde{A}^L})_0$ for the closed fibre of $X_{\tilde{A}^L}$, $X_{\tilde{A}^L} \to X$ induces an isomorphism $(X_{\tilde{A}^L})_0 \tilde{\to} X_0$, whence the canonical profinite group isomorphisms:

$$\pi_1(X_{\tilde{A}^L}, x^L_{(0)}) \tilde{\to} \pi_1((X_{\tilde{A}^L})_0, x^L_0) \tilde{\to} \pi_1(X_0, x_0) \tilde{\to} \pi_1(X, x_{(0)}).$$

So, the problem can now be reformulated as:

Problem: *When does there exist a finite separable extension $K_1 \hookrightarrow K_2$ and $f : Y \to X_{\tilde{A}^{K_2}} \in R^{et}_{X_{\tilde{A}^{K_2}}}$ Galois such that:*

$$\begin{tikzcd} Y_{1K_2} \arrow[r] \arrow[d, "f_{1K_2}"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] & ? \\ X_{K_2} \arrow[r] & X_{\tilde{A}^{K_2}} \end{tikzcd}$$

For any finite extension L/K, $f_{\tilde{A}^L} : X_{\tilde{A}^L} \to \operatorname{spec}(\tilde{A}^L)$ is smooth so $X_{\tilde{A}^L}$ is regular. In particular, since $(X_{\tilde{A}^L})_0$ is an irreducible (normal and connected) closed subscheme of codimension 1 (flatness), its local ring $\mathcal{O}_{X,L} := \mathcal{O}_{X_{\tilde{A}L},(X_{\tilde{A}L})_0}$ is a discrete valuation ring with fraction field L and residue field $k(X_0)$ (hence of characteristic p). Also, $X_{\tilde{A}L}$ being regular, hence normal, and connected is also irreducible thus let R_L denote its ring of rational functions. But, then, it follows from theorem 3.25 (and theorem 2.10) that finite etale Galois covers $f : Y \to X_{\tilde{A}L} \in R^{et}_{X_{\tilde{A}L}}$ correspond to finite Galois field extensions $R_L \hookrightarrow S_L \in FSA_{R_L}$ unramified over $\mathcal{O}_{X,L}$.

With these notation, let u_1 be a uniformizing paramater for $\tilde{A}^{K_1}$ hence for $\mathcal{O}_{X,K_1}$ and $n \geq 1$ a prime-to-p integer. Set $K_2 := K_1[T]/ < T^n - u > /K_1$ then $R_{K_2} = R_{K_1} \otimes_{K_1} K_2 = R_{K_1}[T]/ < T^n - u_1 >$ and, in particular $R_{K_1} \hookrightarrow R_{K_2}$ is tamely ramified over $\mathcal{O}_{X,K_1}$. If we assume that G is of prime-to-p order then $R_{K_1} \hookrightarrow S_{K_1}$ is also tamely ramified over $\mathcal{O}_{X,K_1}$. So, if we choose $n \geq 1$ prime-to-p and multiple of the order of the inertia group of $R_{K_1} \hookrightarrow S_{K_1}$ over $\mathcal{O}_{X,K_1}$ (*e.g.* $n = |G|$) then it follows from lemma 3.29 (3) that $R_{K_2} \hookrightarrow S_{K_2} = S_{K_1} \otimes_{R_{K_1}} R_{K_2}$ is unramified over $\mathcal{O}_{X,K_2}$ (in other words, $f_{1K_2} : Y_{1K_2} \to X_{K_2}$ extends to a finite etale Galois cover $f : Y \to X_{\tilde{A}^{K_2}} \in R^{et}_{X_{\tilde{A}^{K_2}}}$. $\square$

3.4 Fundamental groups of curves: a short review.

Let k be a field of characteristic $p \geq 0$ and let X be a geometrically connected curve over k. Fix a geometric point $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{k}}$ with image $x : \mathrm{spec}(\Omega) \to X$ and $s : \mathrm{spec}(\Omega) \to \mathrm{spec}(k)$.Then the structural morphism $X \to k$ induces the canonical short exact sequence of profinite groups (2):

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, x) \to \pi_1(\mathrm{spec}(k), s) \to 1.$$

Any point in $X(k)$ produces (a conjugacy class of) splitting(s) of (2) but, even if $X(k) = \emptyset$, one has a well-defined action $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(X_{\overline{k}}, \overline{x}))$.

If X is normal and $\overline{x}$, x are geometric generic points, the short exact sequence (2) can be rewritten in terms of usual Galois groups. Indeed, let $M_X/k(X)$ (resp. $M_{X_{\overline{k}}}/\overline{k}(X)$) denote the maximal algebraic extension of $k(X)$ (resp. $\overline{k}(X)$) unramified over X (resp. $X_{\overline{k}}$) in Ω. Then (2) becomes:

$$1 \to \mathrm{Gal}(M_{X_{\overline{k}}}|\overline{k}(X)) \to \mathrm{Gal}(M_X|k(X)) \to \Gamma_k \to 1.$$

So, to understand the "arithmetic fundamental group" $\pi_1(X)$, one should first try to describe the "geometric dundamental group" $\pi_1(X_{\overline{k}})$ and the outer Galois representation $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(X_{\overline{k}}))$. We sum-up below the main classical results about these when X is a smooth, geometrically connected k-curve X of type (g, r).

3.4.1 Proper curves

As already mentioned in step 2 of the proof of theorem 3.22,
- If $\mathrm{char}(k) = 0$ then one has a profinite group isomorphism $\hat{\Gamma}_{g,0} \tilde{\to} \pi_1(X_{\overline{k}})$;
- If $\mathrm{char}(k) = p > 0$ then one has a profinite group epimorphism $\hat{\Gamma}_{g,0} \twoheadrightarrow \pi_1(X_{\overline{k}})$, which, according to theorem 3.31, induces an isomorphism on the prime-to-p completions $\hat{\Gamma}^{(p)'}_{g,0} \tilde{\to} \pi_1(X_{\overline{k}})^{(p)'}$.

These results extend to non necessarily proper curves of type (g, r).

3.5 Curves of type (g, r)

A smooth, geometrically connected k-curve X is said to be *of type* (g, r) if, writing $X \hookrightarrow \tilde{X}$ for the smooth compactification of X, g is the genus of $\tilde{X}$ and r is the degree over k of the reduced divisor $D_X := \tilde{X} \setminus X$. A k-curve X of type (g, r) is said to be *hyperbolic* if it has Euler characteristic

$2-2g-r<0$ (that is $(g,r)\neq(0,0),(0,1),(0,2),(1,0)$) or, equivalently, if $\mathrm{Aut}_{Sch/\overline{k}}(X_{\overline{k}})$ is finite.

We sum up below, without proof, the main statements about smooth, geometrically connected k-curves of type (g,r). For proofs and extension to higner dimensional schemes, we refer to [GM71].

Write $D_X(\overline{k})=\{t_1,\dots,t_r\}$.

1. $\underline{p=0.}$ As X is of finite type over k, one can assume that k is finitely generated over $\mathbb{Q}$ and any fields embedding $k\hookrightarrow\mathbb{C}$ induces an equivalence of categories $R^{et}_{X_{\overline{k}}}\approx R^{et}_{X_{\mathbb{C}}}$. From Rieman existence theorem, one gets an equivalence of categories $R^{et}_{X_{\overline{k}}}\approx R^{et}_{X^{an}_{\mathbb{C}}}$ whence a canonical profinite group isomorphism:
$$\phi:\widehat{\Gamma}_{g,r}=\widehat{\pi_1(X^{an}_{\mathbb{C}})}^{top}\tilde{\to}\pi_1(X_{\overline{k}}).$$
Fix a compatible system $(\zeta_n)_{n\geq 0}$ of primitive nth roots of unity in $\overline{k}$ (that is such that $\zeta^n_{nm}=\zeta_m$, $n,m\geq 0$) and write $M_{X_{\overline{k}}}$ as an inductive limit $M_{X_{\overline{k}}}=\cup_{n\geq 0}M_n$ of finite Galois subextensions of $M_{X_{\overline{k}}}/\overline{k}(X)$. Also, fix a compatible system $(t_{i,n})_{n\geq 0}$ of places of M_n above t_i and a compatible system $(u_{i,n})_{n\geq 0}$ of uniformizing parameters of the $(t_{i,n})_{n\geq 0}$ (that is such that $u^m_{i,nm}=u_{i,n}$, $n,m\geq 0$). Then, for each $n\geq 0$, one gets a canonical (well-defined) group monomorphism $I_{t_{i,n}}\hookrightarrow\overline{k}^{\times}$, $\omega\mapsto\frac{\omega(u_{i,n})}{u_{i,n}}$ mod $t_{i,n}$, where $I_{t_{i,n}}$ denotes the inertia group of $t_{i,n}$ in $M_n/\overline{k}(X)$. The *distinguished generator of the inertia* above t_i in $\pi_1(x_{\overline{k}},\overline{x})$ associated with these data is the inverse image of $(\zeta_{|I_{t_{i,n}}|})_{n\geq 0}$ *via* the canonical morphism $\varprojlim I_{t_{i,n}}\hookrightarrow\overline{k}^{\times}$. These describe a whole conjugacy class W_{t_i} in $\pi_1(x_{\overline{k}},\overline{a})$, called the *inertia canonical class*, when the data $(\zeta_n)_{n\geq 0}$ and $(t_{i,n})_{n\geq 0}$ vary.
Then ϕ sends the generator γ_i of $\widehat{\Gamma}_{g,r}$ to a distinguished generator ω_{t_i} of the inertia group I_{t_i} of t_i in $M_{X_{\overline{k}}}/\overline{k}(X)$, $i=1,\dots,r$. Furthermore,

Lemma 3.32 (Branch cycle argument) $\rho:\Gamma_k\to\mathrm{Out}(\pi_1(X_{\overline{k}},\overline{x}))$ *acts on the inertia canonical class as follows. For any* $\sigma\in\Gamma_k$ $\rho(\sigma)(W_{t_i})=W^{\chi(\sigma)}_{\sigma(t_i)}$, *where* $\chi:\Gamma_k\to\hat{\mathbb{Z}}$ *denotes the cyclotomic character.*

Note that $\widehat{\Gamma}_{0,0}=\widehat{\Gamma}_{0,1}=1$, $\widehat{\Gamma}_{0,2}=\widehat{\mathbb{Z}}$, $\Gamma_{1,0}=\widehat{\mathbb{Z}}^2$ and for any $(g,r)\neq(0,0),(0,1),(0,2),(1,0)$, $\widehat{\Gamma}_{g,r}$ is non abelian. Hence X is hyperbolic af and only if $\pi_1(X_{\overline{k}},\overline{x})$ is non-abelian.

2. $\underline{p>0.}$ The category $R^{D_X}_X$ of all finite covers of $\tilde{X}$ etale over X and tamely ramified over D_X with fiber functors defined by geometric points $x\in\tilde{X}$ Galois with corresponding fundamental group the so-called *tame fundamental group* $\pi_1^{D_X}(X,x)$. Note that, by definition, $\pi_1^{D_X}(X,a)$ is the maximal quotient of $\pi_1(X,a)$ classifying finite covers of $\tilde{X}$ etale over X and tamely ramified over D_X. If x is a geometric generic point on $\tilde{X}$ then $\pi_1^{D_X}(X,a)$ is just the Galois group of the maximal algebraic extension $M_X^{D_X}/k(X)$ unramified over X and tamely ramified over D_X. In particular, one again gets the short exact sequence of finite groups:
$$1\to\pi_1^{D_X}(X_{\overline{k}},\overline{x})\to\pi_1^{D_X}(X,x)\to\pi_1(\mathrm{spec}(k),s)=\Gamma_k\to 1$$
and a well-defined action $\rho:\Gamma_k\to\mathrm{Out}(\pi_1^{D_X}(X_{\overline{k}},\overline{x}))$.

Theorem 3.33 (Pro-p completion of the fundamental group) *Let $r_X := \dim_{\mathbb{F}_p} \mathrm{Jac}_{\tilde{X}|k}[p] \leq g$ denotes the p-rank of the jacobian of $\tilde{X}$. Then*
(i) If $r = 0$ then $\pi_1^{(p)}(X_{\overline{k}}, \overline{x})$ is a free pro-p group on r_X generators.
(ii) If $r > 0$ then $\pi_1^{(p)}(X_{\overline{k}}, \overline{x})$ is a free pro-p group on $|\overline{k}|$ generators.

The theory of specialization also works along the same guidelines. More precisely,there exists a smooth $W(k)$-curve $W(\tilde{X})$ and a divisor $W(D_X) \subset W(\tilde{X})$ etale on $W(k)$ such that, with $W(X) := W(\tilde{X}) \setminus W(D_X)$, one has cartesian squares

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W(\tilde{X}) \arrow[d] \\ k \arrow[r] & W(k) \end{tikzcd} , \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W(X) \arrow[d] \\ k \arrow[r] & W(k) \end{tikzcd}$$

The cartesian squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W(X) \arrow[d] \arrow[dr, phantom, "\square"] & W(X)_1 \arrow[l] \arrow[d] \\ k \arrow[r] & W(k) & K \arrow[l] \end{tikzcd}$$

give rise to a commutative diagram of profinite groups

$$\begin{tikzcd} \pi_1^{D_X}(X) \arrow[r] \arrow[d] & \pi_1^{D_X}(W(X)) \arrow[d] & \pi_1(X_1) \arrow[l] \arrow[d] \\ \Gamma_k \arrow[r] & \pi_1^t(\mathrm{spec}(W(k))) & \Gamma_K, \arrow[l] \end{tikzcd}$$

where $\pi_1^{D_X}(W(X))$ is the maximal quotient of $\pi_1(W(\tilde{X}))$ classifying finite covers of $W(\tilde{X})$ unramified everywhere except over $W(D_X)$ and the generic point of $\tilde{X}$ and $\pi_1^t(\mathrm{spec}(W(k)))$ is the Galois group of the maximal algebraic extension $K \hookrightarrow K^t$ tamely ramified over $W(k)$.

Theorem 3.34 (specialization of tame fundamental groups) *Let $W(k)^t$ denote the extension of $W(k)$ to K^t, then the canonical profinite groups morphism*

$$\pi_1^{D_X}(W(X)_{\overline{K}}) \twoheadrightarrow \pi_1^{D_X}(W(X)_{W(k)^t})$$

is an epimorphism and the canonical group morphism

$$\pi_1^{D_X}(X_{\overline{k}}) \tilde{\rightarrow} \pi_1^{D_X}(W(X)_{W(k)^t}, W(a))$$

is an isomorphism, which yield a well-defined specialization epimorphism:

$$sp : \pi_1^{D_X}(W(X)_{\overline{K}}) \twoheadrightarrow \pi_1^{D_X}(X_{\overline{k}}).$$

Furthermore, $sp : \pi_1^{D_X}(W(X)_{\overline{K}}) \twoheadrightarrow \pi_1^{D_X}(X_{\overline{k}})$ induces a profinite group isomorphism on the prime-to-p completions:

$$sp^{(p)'} : \pi_1^{(p)'}(W(X)_{\overline{K}})) \tilde{\rightarrow} \pi_1^{(p)'}(X_{\overline{k}}).$$

As a result, one gets a canonical profinite group epimorphism

$$\phi : \widehat{\Gamma}_{g,r} \twoheadrightarrow \pi_1^{D_X}(X_{\overline{k}}),$$

such that for each $i = 1, \ldots, r$, ϕ sends the generator γ_i of $\widehat{\Gamma}_{g,r}$ to a distinguished generator ω_{t_i} of the inertia group I_{t_i} above t_i and $\rho^{D_X}(\sigma)(W_{t_i}) = W_{\sigma(t_i)}^{\chi(\sigma)}$, $\sigma \in \Gamma_k$.

Remark 3.35 The only known proof of the above results is *via* Riemann Existence Theorem hence resorts to transcendental methods. Finding an algebraic proof remains a widely open question. In [BE08], such an algebraic proof, relying on Grothendieck-Ogg-Shafarevich formula, is provided for the maximal prime-to-p solvable quotient$\pi_1^{(p'),res}(X_{\overline{k}}, \overline{x})$ of $\pi_1(X_{\overline{k}}, \overline{x})$.

Remark 3.36 From the above, the following information can be read out of $\pi_1(X_{\overline{k}})$:

1. p, except if $X = \mathbb{P}^1_k$. Indeed, $p = 0$ if and only if for any prime l $\pi_1(X_{\overline{k}})^{(l)}$ is of finite type and $p > 0$ if and only if there exists a prime l such that $\pi_1(X_{\overline{k}})^{(l)}$ is not of finite type, in which case $l = p$.

2. Whether X is affine or proper. Indeed, X is proper if and only if $\pi_1(X_{\overline{k}})^{(p)}$ is of finite type.

3. If X is complete then $2g = \mathrm{rank}((\pi_1(X_{\overline{k}})^{(l)})^{ab})$, $l \neq p$ and $r_X = \mathrm{rank}((\pi_1(X_{\overline{k}})^{(p)})^{ab})$.

The general idea of Grothendieck's anabelian geometry is that, considering the arithmetic fundamental group $\pi_1(X)$ instead of the geometric fundamental group $\pi_1(X_{\overline{k}})$, one should be able to recover much more information about X, up to reconstruct its isomorphism class up to canonical ismorphisms.

4 Anabelian geometry - a tentative of definition

The idea of Grothendieck's anabelian geometry is that, provided they satisfy some "anabelian" conditions, geometry and arithmetic of schemes should be encoded in their fundamental group. Though there is no clear definition of what "anabelian" conditions are or of what "being encoded in its fundamental group" means, we will try and make these ideas more explicit.

Note that this section is (at least currently), rather a catalogous of classical anabelian conjectures and results but does not contain any proofs (nor even sketches of). However, the reader interested in going further can consult the comprehensive lecture notes of F. Pop for the A.W.S. 2005 [P05].

4.1 Anabelian categories

In the following, given a profinite group G, we will write

$$\begin{array}{rccl} i: & G & \to & \mathrm{Aut}(G) \\ & g & \mapsto & i(g) = g \cdot - \cdot g^{-1} \end{array}$$

for the inner conjugation morphism, $\mathrm{Inn}(G)$ and $\mathrm{Out}(G)$ for the image and cokernel of i respectively. Recall that $\ker(i) = Z(G)$ is just the center of G.

Also, given a category $\mathcal{C}$, we will write $Gr\mathcal{C}$ for the associated groupoid that is $\mathrm{Ob}(Gr\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{Gr\mathcal{C}} = \mathrm{Isom}_{\mathcal{C}}$.

Let S be a connected scheme. Write Sch/S for the category of S-schemes with dominant morphisms and $Sch^0/S \subset Sch/S$ for the full subcategory of connected objects. The theory of fundamental groups exposed in the preceding sections motivates the introduction of the following category $\mathcal{G}_S$ defined by:

- Objects: pairs (G, π), where G is a profinite group and $\pi : G \to \pi_1(S)$ is a profinite group morphism. Write $\overline{G}$ for the kernel of $\pi : G \to \pi_1(S)$.

- Morphisms: Given two objects (G_i, π_i), $i = 1, 2$ in $\mathcal{G}_S$, the set of morphisms from (G_1, π_1) to (G_2, π_2) in $\mathcal{G}_S$ is the set $I/\sim$, where I is the set of all open profinite group morphisms $\phi : G_1 \to G_2$ such that there exists $\gamma_\phi \in \pi_1(S)$ with $\pi_2 \circ \phi = i(\gamma_\phi) \circ \pi_1$. Then $\mathrm{Inn}(\overline{G}_2)$ acts naturally on the right on I *via* $I \times \mathrm{Inn}(\overline{G}_2) \to I$, $(\phi, i(\overline{g}_2)) \mapsto i(\overline{g}_2) \circ \phi$ and one sets $I/\sim := I/\cdot \mathrm{Inn}(\overline{G}_2)$.

Then, the etale fundamental group functor $\pi_1(-)$ induces natural functors:

$$A_S : Sch^0/S \to \mathcal{G}_S,\ GrA_S : GrSch^0/S \to Gr\mathcal{G}_S$$

With these notations, one can make an attempt to define anabelian categories. A full subcategory $\mathcal{A}_S \subset Sch^0/S$ is said to be *S-Hom-anabelian* (resp. *S-Isom-anabelian*) if $A_S : \mathcal{A}_S \to \mathcal{G}_S$ (resp. if $GrA_S : Gr\mathcal{A}_S \to Gr\mathcal{G}_S$) is fully faithfull. When $\mathcal{A}_S$ has a single object X, we say that X is *S-Hom-anabelian* or *S-Isom-anabelian* if $\mathcal{A}_S$ is.

Requiring the full faithfullness of A_S or even GrA_S might be too much and one might be led to consider weaker notions. Note that A_S induces a set-theoretical map at the level of isomorphism classes of objects:

$$\mathfrak{A}_S : \mathrm{Ob}(\mathcal{A}_S)/\mathrm{Isom}_{\mathcal{A}_S} \to \mathrm{Ob}(\mathcal{G}_S)/\mathrm{Isom}_{\mathcal{G}_S}.$$

With this notation, a full subcategory $\mathcal{A}_S \subset Sch^0/S$ is said to be *S-wIsom-anabelian* (resp. *S-wwIsom-anabelian*) if $\mathfrak{A}_S : \mathrm{Ob}(\mathcal{A}_S)/\mathrm{Isom}_{\mathcal{A}_S} \to \mathrm{Ob}(\mathcal{G}_S)/\mathrm{Isom}_{\mathcal{G}_S}$ is injective (resp. has finite fibers).

Remark 4.1

1. Let us mention two other possible variants for the definition of $\mathcal{G}_S$ and A_S.

   (a) Birational variant: Replace $\pi_1(S)$ with $\Gamma_{k(S)}$ in the definition of $\mathcal{G}_S$ and define A_S to be the relative function field functor sending $X \to S$ to $k(S) \hookrightarrow k(X)$.

   (b) Tame variants: For instance, if $S = \mathrm{spec}(k)$, with k a field of characteristic $p > 0$, replace the fundamenta group functor $\pi_1(-)$ with the tame fundamental group functor $\pi_1^t(-)$ in the definition of A_S. We leave it as an exercise to the reader to generalize those variants for more general notions of tame fundamental groups.

2. Note that most of te above formalism can be extended to any Galois category (See proposition **??**).

4.2 Examples and historical conjectures

4.2.1 Non anabelian categories

So far, most of the examples we considered are NOT anabelian. For instance, if $S = \mathrm{spec}(\mathbb{C})$, Riemann existence theorem roughly tells us that full subcategories $\mathcal{A}_\mathbb{C} \subset Sch^{0TF}/\mathbb{C}$ are far from being anabelian since the fundamental group of their objects encodes no more than topological data of the associated complex analytic space. If we replace $\mathbb{C}$ by any algebraically closed field k of characteristic 0, lemma 3.21 tells us that this remains true (at least) for full subcategories $\mathcal{A}_k \subset Sch^{0TF}/k$ (of proper k-schemes). For instance, if $\mathcal{A}_k$ is the category of all d-dimensional abelian varieties over k then the image of $\mathfrak{A}_k$ consists of a single element, namely $\widehat{\mathbb{Z}}^{2d}$. Similarly, if $\mathcal{A}_k$ is the category of all genus g smooth proper curves over k then the image of $\mathfrak{A}_k$ consists only of $\widehat{\Gamma}_{g,0}$ *etc.* If we replace $\mathbb{C}$ by any algebraically closed field k of characteristic $p > 0$, things already change drastically (see section 4.3), which, in particular, shows the p-part of the fundamental group is a rich invariant over such fields.

Riemann Existence theorem and the specialization theory of fundamental group suggest that one should search for anabelian categories among arithmetic ones. This was, actually, the original intuition of Grothendieck. Another motivation comes from Tate conjecture and side results proved in [F83]. Indeed, let C_i be a smooth projective curve over k; recall that $\pi_1(C_i)^{ab} = \prod_l T_l(Pic(C_i|k))$ as $\widehat{\mathbb{Z}}[\Gamma_k]$-modules, $i = 1, 2$. So, if $\pi_1(C_1)^{ab}$ and $\pi_1(C_2)^{ab}$ are Γ_k-isomorphic then $Pic(C_1|k)$ and $Pic(C_2|k)$ are isogenous over k and, fixing C_2, there are only finitely many possibilities for the isomorphism class of $Pic(C_1|k)$ over k (isogeny theorem) hence for the isomorphism class of C_1 over k (Torelli's theorem). This already shows that the category $\mathcal{A}_k$ of all smooth projective curves over k is k-wwIsom-anabelian. But the abelianization $\pi_1(X)^{ab}$ of the fundamental group viewed as a Γ_k-module encodes much less than the datum of $\pi_1(X) \to \Gamma_k$, which one can expect to be rich enough to narrow the number of smooth projective curves over k with the same fundamental group from "finitely many" to "one".

4.2.2 Grothendieck's examples of categories that should be anabelian

Let k be a finitely generated field (over its prime field). Then, the following categories are expected to be k-anabelian[5]:
- $\mathcal{A}_k$ = category of all 0-dimensional connected schemes of finite type over k.
- $\mathcal{A}_k$ = category of all smooth, geometrically connected and hyperbolic curves over k. Recall that a curve X over k is said to be hyperbolic if, writing $X \hookrightarrow \tilde{X}$ for the smooth compactification of X, g for the genus of $\tilde{X}$ and r for $|\tilde{X} \setminus X|$ then $2 - 2g - r < 0$ (that is $(g, r) \neq (0, 0), (0, 1), (0, 2), (1, 0)$ or, equivalently, $\mathrm{Aut}_{\overline{k}}(X)$ is finite).
- $\mathcal{A}_k$ = category of all elementary anabelian schemes over k. A scheme X over k is said to be an *elementary anabelian scheme over k* if there exists a finite sequence of k-morphisms $X = X_0 \to X_1 \to \cdots \to X_n = \mathrm{spec}(k)$ with $X_{i-1} \to X_i$ a relative hyperbolic curve, $i = 1, \dots, n$.

Eventually, the moduli schemes $M_{g,r}$ $(2-2g-r < 0)$ and $A_{g,d}$ are also expected to be $\mathbb{Z}$-anabelian. Apart from these, there does not seem to have clearly stated anabelian conjectures in higher (*i.e.* ≥ 2) dimensions. (Recall corollary 3.27 and example 3.28).

4.2.3 The section conjecture

Let k be a field and X a normal scheme geometrically connected over k. Then one has the canonical short exact sequence from Galois theory:

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k \to 1 \tag{9}$$

Now, any k-rational point $x : \mathrm{spec}(k) \to X$ induces a ($\pi_1(X_{\overline{k}})$-conjugacy class of) section morphism(s) $s_x : \Gamma_k \to \pi_1(X)$ splitting (9). The $\pi_1(X_{\overline{k}})$-conjugacy classes of section morphisms $s : \Gamma_k \to \pi_1(X)$ splitting (9) can be regarded as $\mathrm{hom}_{\mathcal{G}_k}(\Gamma_k, \pi_1(X))$ and a special case of the section conjecture asserts that if k is a finitely generated field of characteristic 0 and if X is a proper hyperbolic curve over k then the following canonical map is bijective:

$$\mathrm{Hom}_{Sch/k}(\mathrm{spec}(k), X) \tilde{\to} \mathrm{Hom}_{\mathcal{G}_k}(\Gamma_k, \pi_1(X)). \tag{10}$$

[5] At this level, we do not give a precise meaning to "k-anabelian" since Grothendieck himself remained vague in his formulations.

More generally, let X be a hyperbolic curve of type (g,r) over k and let $X \hookrightarrow \tilde{X}$ be its smooth compactification. Then, for any k-rational point $\tilde{x} \in \tilde{X}(k)$, the short exact sequence

$$1 \to I_{\Gamma_{k(X)}}(\tilde{x}) \to D_{\Gamma_{k(X)}}(\tilde{x}) \to \Gamma_k \to 1 \tag{11}$$

always splits but this splitting is not unique up to inner conjugation by elements of $\Gamma_{\overline{k}(X)}$ hence, if $\tilde{x} \in \tilde{X}(k) \setminus X(k)$, $\tilde{x}$ gives rise to several $\pi_1(X_{\overline{k}})$-conjugacy classes of splitting sections of (9) and the map (10) does not extend *a priori* to $\hom_{Sch/k}(\mathrm{spec}(k), \tilde{X})$. A splitting section $s : \Gamma_k \to \pi_1(X)$ of (9) is said to be *unbranched* if $s(\Gamma_k)$ is contained in no decomposition group of a point $\tilde{x} \in \tilde{X}(k) \setminus X(k)$ in $\pi_1(X)$. A basic form of the section conjecture can thus be formulated as follows:

Conjecture 4.2 (Section conjecture) *Let k be a number field and let X be a hyperbolic curve over k. Then the canonical map $\mathrm{Hom}_{Sch/k}(\mathrm{spec}(k), X) \tilde{\to} \mathrm{Hom}_{\mathcal{G}_k}(\Gamma_k, \pi_1(X))$ is injective and induces a bijection onto the set of $\pi_1(X_{\overline{k}})$-conjugacy classes of unbranched sections. Furthermore, any splitting section $s : \Gamma_k \to \pi_1(X)$ of (9) arises geometrically (i.e. is induced by a splitting of a short exact sequence like (11)).*

4.3 Results

4.3.1 0-dimensional case

Theorem 4.3 (Artin-Schreier) *Let k be a field with non trivial finite absolute Galois group Γ_k. Then $\Gamma_k = \mathbb{Z}/2$ and k is real-closed (in particular, k has characteristic 0 and $\overline{k} = k(\sqrt{-1})$).*

The Artin-Schreier theorem shows that the assumption that the absolute Galois group is non trivial finite already imposes restrictions on k but these are not on the isomorphism type of k (there are infinitely many isomorphism classes of real closed field and their classification seems to be currenly out of reach). However, if $k \subset \overline{\mathbb{Q}}$ is a field of algebraic numbers, Artin-Schreier theorem shows that if k has non trivial finite absolute Galois group then k is isomorphic to $\mathbb{R} \cap \overline{\mathbb{Q}}$. In particular, the subextension of $\overline{\mathbb{Q}}$ with non trivial finite absolute Galois group are exactly the $\sigma(\mathbb{R} \cap \overline{\mathbb{Q}})$, $\sigma \in \Gamma_{\mathbb{Q}}$.

Later, Neukirch proved a p-adic analog of the Artin-Schreier theorem, which was the first main step towards the proof of the 0-dimensional anabelian conjectures.

Theorem 4.4 (Neukirch) *Let $k \subset \overline{\mathbb{Q}}$ be subfield and p a prime number.*
(i) Assume that $\Gamma_k \simeq \Gamma_{\mathbb{Q}_p}$. Then there exists a place $P \in \mathcal{P}(\overline{\mathbb{Q}})$ such that k is the decomposition field of P in $\overline{\mathbb{Q}}/\mathbb{Q}$.
(ii) Assume that Γ_k is isomorphic (as profinite group) to an open subgroup of $\Gamma_{\mathbb{Q}_p}$. Then there exists a place $P \in \mathcal{P}(\overline{\mathbb{Q}})$ such that k is a finite extension of the decomposition field of P in $\overline{\mathbb{Q}}/\mathbb{Q}$.

A consequence of Neukirch theorem is that, given two number fields k_1, k_2, any profinite group isomorphism $\Phi : \Gamma_{k_1} \tilde{\to} \Gamma_{k_2}$ induces an arithemetical equivalence $\phi : \mathcal{P}(k_1) \tilde{\to} \mathcal{P}(k_2)$ that is a bijection between the places of $k1$ and thowe of k_2 preserving the invatiants $e(P|p)$ and $f(P|p)$. this, in turn, implies that if $k_1/\mathbb{Q}$ is Galois then k_1 and k_2 are isomorphic a fields hence, automatically, as $\mathbb{Q}$-extensions. But since $k_1/\mathbb{Q}$ is norma, one gets $k_1 = k_2$ as subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$ hence $\Gamma_{k_1} = \Gamma_{k_2}$ as subgroups of $\Gamma_{\mathbb{Q}}$. In particular, this shows that any open normal subgroup of $\Gamma_{\mathbb{Q}}$ is characteristic. This lead Neukirch to ask:

- Do we have $\mathrm{Aut}(\Gamma_{\mathbb{Q}}) = \mathrm{Inn}(\Gamma_{\mathbb{Q}})$?
- Is any profinite group isomorphism $\Phi : \Gamma_{k_1} \tilde{\to} \Gamma_{k_2}$ as above induced by inner conjugation by an element $\sigma \in \Gamma_{\mathbb{Q}}$?

Both answers were anwered positively in the 70's to get:

Theorem 4.5 (Neukirch-Ikeda-Iwasawa-Uchida) *Let k_1, k_2 be global fields. The nfor any profinite group isomorphism $\Phi : \Gamma_{k_1} \tilde{\to} \Gamma_{k_2}$ there exists a unique field isomorphism $\phi : k_1^s \tilde{\to} k_2^s$ such that $\Phi(g_1) = \phi^{-1} g_1 \phi$, $g_1 \in \Gamma_{k_1}$. In particular,*
- *$\phi(k_1) = k_2$.*
- *$\mathrm{Isom}(k_1, k_2) \tilde{\to} \mathrm{Isom}_{\mathcal{G}_{\mathbb{Z}}}(\Gamma_{k_1}, \Gamma_{k_2}) (= \mathrm{Out}(\Gamma_{k_1}, \Gamma_{k_2}))$.*

Given a field k, let k^i denote its purely inseparable closure. Also, two fields morphisms ϕ, $\psi : k_1 \to k_2$ are said to be quivalent up to Frobenius twist if $\psi = \phi \circ Fr^n$, where $Fr : k_1 \tilde{\to} k_1, x \mapsto x^{\mathrm{char}(k_1)}$ denotes the absolute Frobenius on k_1. With these notation, let us give a more precise formualtion of the 0-dimensional anabelian conjectures:

Conjecture 4.6 (0-dimensional anabelian conjectures)

1. *Absolute forms:*
   - (a) 0-dimensional $\mathbb{Z}$-Isom-anabelian conjecture: *Given any finitely generated infinite fields k_1, k_2 and any profinite group isomorphism $\Phi : \Gamma_{k_1} \tilde{\to} \Gamma_{k_2}$ there exists a field isomorphism $\phi : \overline{k}_2 \tilde{\to} \overline{k}_1$, unique up to Frobenius twist, and such that $\Phi(g_1) = \phi^{-1} \circ g_1 \circ \phi$, $g_1 \in \Gamma_{k_1}$ (in particular, $\phi(k_2^i) = k_1^i$).*
   - (b) 0-dimensional $\mathbb{Z}$-Hom-anabelian conjecture: *Given any finitely generated infinite fields k_1, k_2 and any open profinite group morphism $\Phi : \Gamma_{k_1} \to \Gamma_{k_2}$ there exists a field embedding $\phi : \overline{k}_2 \hookrightarrow \overline{k}_1$, unique up to Frobenius twist, and such that $\phi(\overline{k}_2) \subset \phi(\overline{k}_2)$ and $\Phi(g_1) = \phi^{-1} \circ g_1 \circ \phi$, $g_1 \in \Gamma_{k_1}$ (in particular, $\phi(k_2^i) \subset k_1^i$).*
2. *Relative forms: Let k be a field.*
   - (a) 0-dimensional k-Isom-anabelian conjecture: *Given any fields extensions k_1/k, k_2/k and any profinite group Γ_k-isomorphism $\Phi : \Gamma_{k_1} \tilde{\to} \Gamma_{k_2}$ there exists an isomorphism $\phi : \overline{k}_2 \tilde{\to} \overline{k}_1$ of k-extensions, unique up to Frobenius twist, and such that $\Phi(g_1) = \phi^{-1} \circ g_1 \circ \phi$, $g_1 \in \Gamma_{k_1}$ (in particular, $\phi(k_2^i) = k_1^i$).*
   - (b) 0-dimensional k-Hom-anabelian conjecture: *Given any fields extensions k_1/k, k_2/k and any open profinite group Γ_k-morphism $\Phi : \Gamma_{k_1} \to \Gamma_{k_2}$ there exists an embedding $\phi : \overline{k}_2 \hookrightarrow \overline{k}_1$ of k-extensions, unique up to Frobenius twist, and such that $\phi(\overline{k}_2) \subset \phi(\overline{k}_2)$ and $\Phi(g_1) = \phi^{-1} \circ g_1 \circ \phi$, $g_1 \in \Gamma_{k_1}$ (in particular, $\phi(k_2^i) \subset k_1^i$).*

The final proof of the 0-dimensional $\mathbb{Z}$-anabelian conjectures was finally established by Pop (Isom-form) in all characteristics [P94] and Mochizuki (Hom-form) in characteristic 0 [M99].

Theorem 4.7 (0-dimensional anabelian conjectures) *Conjecture4.6 (1) (a) holds in any characteristic and conjecture 4.6 (1) (b) holds in characteristic* 0.

Actually, Mochizuki's result is a consequence of its [M99, Th. B] stating that for any sub-p-adic field k (see section 4.8) conjecture 4.6 (2) (b) holds for regular, finitely generated extensions of k.

4.3.2 1-dimensional case

Given a k-curve $X \to k$, let $X^i \to k^i$ denote the normalization of X in $k(X)^i/k(X)$. Also, write $X(n) := X \times_{k,Fr^n} k \to k$ for the nth Frobenius twist of $X \to k$. With these notation, one can, as in the 0-dimensional cases, give a precise formulation of the 1-dimensional anabelian conjectures.

Conjecture 4.8 (1-dimensional anabelian conjectures)

1. Absolute forms:

   (a) 1-dimensional $\mathbb{Z}$-Isom-anabelian conjecture: *Given any hyperbolic curves $X_1 \to k_1$, $X_2 \to k_2$ defined over finitely generated base fields k_1, k_2 and any profinite group isomorphism $\Phi : \pi_1(X_1)\tilde{\to}\pi_1(X_2)$ there exists a curves isomorphism $\phi : X_1^i\tilde{\to}X_2^i$, unique up to Frobenius twist and such that $\Phi = \pi_1(\phi)$.*

   (b) 1-dimensional $\mathbb{Z}$-Hom-anabelian conjecture: *Given any hyperbolic curves $X_1 \to k_1$, $X_2 \to k_2$ defined over finitely generated base fields k_1, k_2 and any open profinite group morphism $\Phi : \pi_1(X_1) \to \pi_1(X_2)$ there exists a dominant curves morphism $\phi : X_1^i \to X_2^i$, unique up to Frobenius twist and such that $\Phi = \pi_1(\phi)$.*

2. Relative forms: *Let k be a field.*

   (a) 1-dimensional k-Isom-anabelian conjecture: *Given any hyperbolic k-curves X_1, X_2 and any profinite group Γ_k-isomorphism $\Phi : \pi_1(X_1)\tilde{\to}\pi_1(X_2)$ there exists a k-curves isomorphism $\phi : X_1^i\tilde{\to}X_2^i$, unique up to Frobenius twist and such that $\Phi = \pi_1(\phi)$.*

   (b) 1-dimensional k-Hom-anabelian conjecture: *Given any hyperbolic k-curves X_1, X_2 and any open profinite group Γ_k-morphism $\Phi : \pi_1(X_1) \to \pi_1(X_2)$ there exists a dominant k-curves morphism $\phi : X_1^i \to X_2^i$, unique up to Frobenius twist and such that $\Phi = \pi_1(\phi)$.*

We list below the main "classical results" (quoting Akio Tamagawa) about 1-dimensional anbelian conjectures together with the original references.

1. A. Tamagawa [T97]:

   (a) Affine (hyperbolic) curves over finite fields.
   (i) Conjecture 4.8 (1) (a) (resp. the tame variant of conjecture 4.8 (1) (a)) holds for affine (resp. affine hyperbolic) curves over finite fields.
   (ii) Given a finite field k, conjecture 4.8 (2) (a) (resp. the tame variant of conjecture 4.8 (2) (a)) holds for affine (resp. affine hyperbolic) k-curves.

   (b) Affine hyperbolic curves over finitely generated fields of characteristic 0.
   (i) Conjecture 4.8 (1) (a) holds for affine hyperbolic curves over finitely generated fields of characteristic 0.
   (ii) Given a finitely generated field k of characteristic 0, conjecture 4.8 (2) (a) holds for affine hyperbolic k-curves.

2. S. Mochizuki:

(a) Hyperbolic curves over finitely generated fields of characteristic 0 [M96]. The results below extend Tamagawa's results (b) to arbitrary hyperbolic curves.
(i) Conjecture 4.8 (1) (a) holds for hyperbolic curves over finitely generated fields of characteristic 0.
(ii) Given a finitely generated field k of characteristic 0, conjecture 4.8 (2) (a) holds for hyperbolic k-curves.

(b) Hyperbolic curves over sub-p-adic fields [M99]. A field k is said to be *sub-p-adic* if it can be embedded into a finitely generated extension of $\mathbb{Q}_p$. Given a sub-p-adic field k and a geometrically connected k-scheme X, set $N^{(p)}(X_{\overline{k}}) := \ker(\pi_1(X_{\overline{k}}) \to \pi_1(X_{\overline{k}})^{(p)})$, where $\pi_1(X_{\overline{k}}) \to \pi_1(X_{\overline{k}})^{(p)}$ denotes the pro-p completion of $\pi_1(X_{\overline{k}})$. Since $N^{(p)}(X_{\overline{k}})$ is characteristic in $\pi_1(X_{\overline{k}})$, it is normal in $\pi_1(X)$ and one can form the quotient $\Pi_X^{(p)} := \pi_1(X_{\overline{k}})/N^{(p)}(X_{\overline{k}})$. With these notation, the short exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k \to 1$$

induces a short exact sequence:

$$1 \to \pi_1(X_{\overline{k}})^{(p)} \to \Pi_X^{(p)} \to \Gamma_k \to 1 \tag{12}$$

and the following strong variant of conjecture 4.8 (2) (b) holds [M99, Th. A]: for any hyperbolic k-curve X the functors $Y \mapsto \mathrm{Hom}_{\mathcal{S}ch/k}(Y, X)$ and $Y \mapsto \mathrm{Hom}_{\mathcal{G}_k}(\Pi_Y^{(p)}, \Pi_X^{(p)})$ from the category of geometrically integral k-schemes of finite type to sets are isomorphic and this isomorphism is functorial in X.
From this result, Mochizuki derived the following higher dimensional significative result in anabelian geometry [M99, Th. D]. An *hyperbolically fibered surface X over k* is a k-scheme $X = \tilde{X} \setminus D$, where $\tilde{X}$ is a smooth proper hyperbolic curve over an hyperbolic k-curve S and $D \subset \tilde{X}$ is a divisor, etale on S. Then conjecture 4.8 (2) (a) holds for hyperbolically fibered surfaces over k.

3. J. Stix: Let k be a finitely generated field of characteristic $p > 0$.
Recall that a k-curve X is isotrivial if there exists a finite extension F of the base field $\mathbb{F}_p$ such that $X_{\underline{k}}$ is defined over F.

(a) Affine hyperbolic curves over finitely generated fields of characteristic $p > 0$ [S02]. The tame variant of conjecture 4.8 (2) (a) holds for any affine hyperbolic k-curves X_1, X_2 provided one of them is not isotrivial. If X_1, X_2 are two affine isotrivial hyperbolic k-curves, then the the image of $\mathrm{isom}_{\mathcal{S}ch/k}(X_1, X_2) \to \mathrm{isom}_{\mathcal{G}_k}(\pi_1^t(X_1), \pi_1^t(X_2))$ is dense and the uniqueness part of the tame variant of conjecture 4.8 (2) (a) holds.
(b) Hyperbolic curves over finitely generated fields of characteristic $p > 0$ [S02]. Conjecture 4.8 (2) (a) holds for any hyperbolic k-curves X_1, X_2 provided one of them is not isotrivial. If X_1, X_2 are two isotrivial hyperbolic k-curves, then the the image of $\mathrm{isom}_{\mathcal{S}ch/k}(X_1, X_2) \to \mathrm{isom}_{\mathcal{G}_k}(\pi_1(X_1), \pi_1(X_2))$ is dense and the uniqueness part of conjecture 4.8 (2) (a) holds.

4.3.3 Higher dimensional results.

As already mentioned, the picture in dimension ≥ 2 is unclear. So, let us mention only a few results (kindly communicated by Akio Tamagawa during his lectures at the ACAG 2007).

1. Shimura varieties (Y. Ihara, H. Nakamura): Let k be a number field then no Siegel or Hilbert modular variety of dimension ≥ 2 is anabelian (Recall that $A_{g,d}$ was expected to be anabelian).

2. Elementary anabelian schemes:
- S. Mochizuki: The category of elementary anabelian surfaces over sub-p-adic fields is k-Isom-anabelian.
- Given an hyperbolic curve $X \to k$, let $X_n := \{\underline{x} \in X^n \mid x_i \neq x_j,\ 1 \leq i \neq j \leq n\}$ denotes the configuration space for n distinct ordered points on X. Then $X_{n+1} \to X_n \to \cdots \to X_1 = X$ are elementary anabelian schemes and:
– H. Nakamura, A. Tamagawa: if k is a finitely generated field of characteristic 0 then X_n is k-Isom-anabelian for $X = \mathbb{P}^1_k \setminus \{0, 1\infty\}$. This results was next extended by
– S. Mochizuki, H. Nakamura, N. Takao: if k is a finitely generated field of characteristic 0 then X_n is k-anabelian for any hyperbolic curves X.
– S. Mochizuki, A.Tamagawa: if k is a sub-p-adic field then the category $\mathcal{C}_n \subset Sch^{0TF}/k$ of all etale covers of some X_n, for $X \to k$ a proper hyperbolic k-curve is k-Isom-anabelian.

3. Moduli spaces of curves:
- J. Stix: If S is a normal scheme and $U \subset S$ a non-empty open subset then the following canonical diagram is cartesian:

$$\begin{tikzcd} \mathrm{Hom}_{Sch/\mathbb{Z}}(S, M_{g,r}) \arrow[r, "(*)"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Hom}_{\mathcal{G}_\mathbb{Z}}(\pi_1(S), \pi_1(M_{g,r})) \arrow[d] \\ \mathrm{Hom}_{Sch/\mathbb{Z}}(U, M_{g,r}) \arrow[r, "(**)"] & \mathrm{Hom}_{\mathcal{G}_\mathbb{Z}}(\pi_1(U), \pi_1(M_{g,r})). \end{tikzcd}$$

- M. Boggi, P. Lochak[6]: Assume that if $g = 0$, $r \geq 5$, if $g = 1$, $r \geq 3$, if $g = 2$, $r \geq 1$ and if $g \geq 3$, $r \geq 0$. Let k be a sub-p-adic field and $X \to M_{g,rk} \in R^{et}_{M_{g,rk}}$. Then the canonical map:

$$\mathrm{Aut}_{Sch/k}(X) \tilde{\to} \mathrm{Out}^*_{\mathcal{G}_k}(\pi_1(X_{\overline{k}}))$$

is bijective (where $*$ means inertia preserving).

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