Finite irreducible monodromy group for Lauricella's F_C

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Monodromy and Hypergeometric Functions

Introduction

We consider conditions when the monodromy group for F_C is finite irreducible.

Remark

In [Bod (2012)], the algebraicity condition for F_C is obtained from the result [Beukers (2010)] for algebraic A-hypergeometric functions.
In this talk, I will give another approach based on the monodromy group.

Contents

- Lauricella's hypergeometric function F_C, basic facts;
- Fundamental group of the complement of the singular locus;
- Monodromy group and its finiteness condition.

Lauricella's F_C

$$F_C(a,b,c_1,\dots,c_n;x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n} m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n}.$$

$$\left(\begin{array}{c} a, b, c = (c_1,\dots,c_n) : \text{parameters} \\ x = (x_1,\dots,x_n) : \text{variables} \\ (\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1) \end{array} \right)$$

It converges in

$$D_C = \left\{ (x_1,\dots,x_n) \in \mathbb{C}^n \;\middle|\; \sum_{k=1}^{n} \sqrt{|x_k|} < 1 \right\}.$$

In the case of $n=2$, it is also called Appell's F_4.

Lauricella	F_A	F_B	F_C	F_D
Appell	F_2	F_3	F_4	F_1

Lauricella's $F_C(a,b,c;x)$ satisfies differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)]\, f(x) = 0 \quad (k = 1, \ldots, n),$$

where $\partial_k = \dfrac{\partial}{\partial x_k}$, $\theta_k = x_k \partial_k$, $\theta = \theta_1 + \cdots + \theta_n$.
We consider the system $E_C(a,b,c)$ generated by them.

Fact ([Hattori-Takayama (2014)])

(i) rank of $E_C = 2^n$.

(ii) The singular locus is

$$S = \left(\prod_{k=1}^{n} x_k \cdot \prod_{\varepsilon_1,\ldots,\varepsilon_n = \pm 1} \left(1 + \sum_{k=1}^{n} \varepsilon_k \sqrt{x_k}\right) = 0 \right) \subset \mathbb{C}^n.$$

$$\Downarrow$$

$x \in \mathbb{C}^n - S$,
Sol_x : the space of local solutions to $E_C(a,b,c)$ around x
$\Longrightarrow \dim Sol_x = 2^n$.

We set $X = \mathbb{C}^n - S$. Recall that the defining equation of S is

$$\prod_{k=1}^{n} x_k \cdot \prod_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \left(1 + \sum_{k=1}^{n} \varepsilon_k \sqrt{x_k}\right).$$

Note that

$$R(x) = \prod_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \left(1 + \sum_{k=1}^{n} \varepsilon_k \sqrt{x_k}\right)$$

is an irreducible polynomial in $x_1, \ldots, x_n$ of degree 2^{n-1}.

We put $S^{(n)} = (R(x) = 0) \subset \mathbb{C}^n$. Then,

$$S = (x_1 = 0) \cup \cdots \cup (x_n = 0) \cup S^{(n)}.$$

$$S^{(n)} = (R(x) = 0), \quad R(x) = \prod_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \left(1 + \sum_{k=1}^{n} \varepsilon_k \sqrt{x_k}\right)$$

$\boxed{n = 1}$ $R(x_1) = (1 - \sqrt{x_1})(1 + \sqrt{x_1}) = 1 - x_1.$

$\boxed{n = 2}$

$$\begin{aligned} R(x_1, x_2) &= (1 - \sqrt{x_1} - \sqrt{x_2})(1 + \sqrt{x_1} - \sqrt{x_2}) \\ &\quad \cdot (1 - \sqrt{x_1} + \sqrt{x_2})(1 + \sqrt{x_1} + \sqrt{x_2}) \\ &= x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1. \end{aligned}$$

$\boxed{n = 3}$

$$\begin{aligned} R(x_1, x_2, x_3) &= (1 - \sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3})(1 + \sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3}) \cdots \\ &= \left(2(x_1^2 + x_2^2 + x_3^2 + 1) - (x_1 + x_2 + x_3 + 1)^2\right)^2 - 64x_1x_2x_3. \end{aligned}$$

If $n \geq 3$, then $S^{(n)}$ has singularities.

$\dot{x} = \left(\frac{1}{2n^2}, \ldots, \frac{1}{2n^2}\right) \in X$: a base point.

Theorem 1 ([G. (2016)],[G.-Kaneko (2018)],[Terasoma; arXiv:1803.06609])

$\pi_1(X, \dot{x})$ is presented as follows.

Generators $n+1$ loops:

ρ_0 : the loop turning the divisor $S^{(n)} = (R(x) = 0)$
around the point $\left(\frac{1}{n^2}, \ldots, \frac{1}{n^2}\right)$,

ρ_k : the loop turning the divisor $(x_k = 0) \quad (1 \leq k \leq n)$.

Relations

(I) $[\rho_i, \rho_j] = 1$ for $1 \leq i, j \leq n$;

(II) $(\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2$ for $1 \leq k \leq n$;

(III) $[(\rho_{i_1} \cdots \rho_{i_p})\rho_0(\rho_{i_1} \cdots \rho_{i_p})^{-1}, (\rho_{j_1} \cdots \rho_{j_q})\rho_0(\rho_{j_1} \cdots \rho_{j_q})^{-1}] = 1$
for $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\} \subset \{1, \ldots, n\}$
with $p, q \geq 1$, $p + q \leq n - 1$ and $I \cap J = \emptyset$.

(I) $[\rho_i, \rho_j] = 1$

(II) $(\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2$

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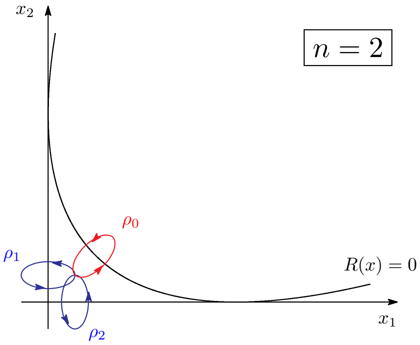
[Kaneko (1981)] $\longrightarrow n = 2$

[G. (2016)]
$\longrightarrow$ the generators
and relations (I), (II)

[G.-Kaneko (2018)]
$\longrightarrow$ relation (III) and $n = 3$

[Terasoma; arXiv:1803.06609]
$\longrightarrow n$: general



I will introduce the proof later.

In [G. (2016)], the circuit transformations along $\rho_0, \rho_1, \ldots, \rho_n$ are studied in the framework of twisted homology group which is naturally isomorphic to $Sol_{\dot{x}}$.

In [G.-Matsumoto (2019)], we construct a basis of $Sol_{\dot{x}}$ under the irreducibility condition. ($c_1, \ldots, c_n \in \mathbb{Z}$ is allowed.)

Fact ([Hattori-Takayama (2014)], [G.-Matsumoto (2019)])

$E_C(a,b,c)$ is irreducible if and only if

$$a - \sum_{k=1}^{n} i_k c_k, \quad b - \sum_{k=1}^{n} i_k c_k \notin \mathbb{Z}, \qquad \forall I = (i_1, \ldots, i_n) \in \{0,1\}^n.$$

By using this basis, we obtain (simple) matrix expressions of the monodromy representation:

$$\mathcal{M} : \pi_1(X, \dot{x}) \to \mathbf{GL}_{2^n}(\mathbb{C}).$$

The monodromy representation

$$\mathcal{M} : \pi_1(X, \dot{x}) \to \mathbf{GL}_{2^n}(\mathbb{C})$$

Today, we consider the monodromy group $\mathbf{Mon} = \mathcal{M}(\pi_1(X, \dot{x}))$.

Since we have $n+1$ generators $\rho_0, \rho_1, \ldots, \rho_n$ of $\pi_1(X, \dot{x})$, the monodromy group $\mathbf{Mon}$ is generated by the circuit matrices

$$M_i = \mathcal{M}(\rho_i) \in \mathbf{GL}_{2^n}(\mathbb{C}) \qquad (i = 0, 1, \ldots, n).$$

The circuit matrices [G.-Matsumoto (2019)]

Notation

We set $\alpha = e^{2\pi\sqrt{-1}a}$, $\beta = e^{2\pi\sqrt{-1}b}$, $\gamma_k = e^{2\pi\sqrt{-1}c_k}$.
We regard $\{0,1\}^n$ as an index set of $\mathbb{C}^{2^n}$.

For $k = 1, \ldots, n$, we have

$$M_k = E_2 \otimes \cdots \otimes E_2 \otimes \underset{k\text{-th}}{\begin{pmatrix} 1 & -\gamma_k^{-1} \\ 0 & \gamma_k^{-1} \end{pmatrix}} \otimes E_2 \otimes \cdots \otimes E_2.$$

The matrix M_0 is written as

$$M_0 = E_{2^n} - {}^t(\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{v}),$$

where $\boldsymbol{v} \in \mathbb{C}^{2^n}$ is a column vector whose I-th entry is

$$\begin{cases} (-1)^n \frac{(\alpha-1)(\beta-1)\prod_{k=1}^n \gamma_k}{\alpha\beta} & (I = (0,\ldots,0)), \\ (-1)^{n+|I|} \frac{(\alpha\beta+(-1)^{|I|}\prod_{k=1}^n \gamma_k^{i_k})\prod_{k=1}^n \gamma_k^{1-i_k}}{\alpha\beta} & (I \neq (0,\ldots,0)). \end{cases}$$

Example ($n = 3$)

M_1, M_2, M_3 (turning $(x_k = 0)$) are upper triangular.

$$M_1 = \begin{pmatrix} 1 & -\gamma_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\gamma_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\gamma_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\gamma_1^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1^{-1} \end{pmatrix}$$

M_0 (turning the hypersurface $S^{(n)}$) is lower triangular.

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & \cdots & & & \cdots & * & \color{red}{*} \end{pmatrix}$$

The last row is

$$\frac{(\alpha-1)(\beta-1)\gamma_1\gamma_2\gamma_3}{\alpha\beta},\ -\frac{(\alpha\beta-\gamma_1)\gamma_2\gamma_3}{\alpha\beta},\ -\frac{(\alpha\beta-\gamma_2)\gamma_1\gamma_3}{\alpha\beta},\ \frac{(\alpha\beta+\gamma_1\gamma_2)\gamma_3}{\alpha\beta},$$
$$-\frac{(\alpha\beta-\gamma_3)\gamma_1\gamma_2}{\alpha\beta}, \frac{(\alpha\beta+\gamma_1\gamma_3)\gamma_2}{\alpha\beta}, \frac{(\alpha\beta+\gamma_2\gamma_3)\gamma_1}{\alpha\beta}, \color{red}{\frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta}}.$$

In general, the eigenvalues of M_0 are

$$\color{red}{\delta_0 = (-1)^{n-1}\gamma_1\cdots\gamma_n\alpha^{-1}\beta^{-1}} \quad \text{and}$$
$$1 \quad (\leftarrow \text{multiplicity} = 2^n - 1).$$

$$\mathbf{Mon} = \langle M_0, M_1, \ldots, M_n \rangle \subset \mathbf{GL}_{2^n}(\mathbb{C})$$

Since the matrices $M_0, M_1, \ldots, M_n$ depend on α, β, $\gamma_1, \ldots, \gamma_n$, we denote

$$\mathbf{Mon} = \mathbf{Mon}^{(n)} = \mathbf{Mon}^{(n)}(\alpha, \beta, \gamma).$$

Note that M_0 is a reflection ($\Leftrightarrow \mathrm{rank}(M_0 - E_{2^n}) = 1$).
We also denote the special eigenvalue of M_0 as

$$\delta_0 = \delta_0^{(n)}(\alpha, \beta, \gamma) = (-1)^{n-1}\gamma_1 \cdots \gamma_n \alpha^{-1}\beta^{-1}.$$

Finite irreducible monodromy group

For Appell's F_4 $(n = 2)$, the following fact has been known.

Fact ([Kato (1997)])

$\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2))$ is finite irreducible if and only if

(1) $\mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_1)$ and $\mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_2)$ are finite irreducible;

(2) (a) at least two of γ_1, γ_2, $\beta\alpha^{-1}$ are -1,
or
(b) $\delta_0^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2)) = -1 \quad (\Leftrightarrow \gamma_1\gamma_2\alpha^{-1}\beta^{-1} = 1)$.

Remark

Since $\mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_i)$ is nothing but the monodromy group of Gauss' HGDE $E(a, b, c_i)$, the conditions for (1) is written in terms of a, b, c in [Schwarz (1873)].

We consider its generalization.

Main theorem

Theorem 2 ([G.; arXiv:1905.00250])

We assume $n \geq 3$.
$\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ is finite irreducible if and only if

(1) each $\mathbf{Mon}^{(1)}(\alpha, \beta, \gamma_k)$ $(k = 1, \ldots, n)$ is finite irreducible;

(2) at least n of $\gamma_1, \ldots, \gamma_n$, $\beta\alpha^{-1}$, $\delta_0^{(n)}(\alpha, \beta, \gamma)$ are -1.

Remark

The condition (2) is divided into

(a) at least n of $\gamma_1, \ldots, \gamma_n$, $\beta\alpha^{-1}$ are -1,

or

(b) $\delta_0^{(n)}(\alpha, \beta, \gamma) = -1$ and
at least $n - 1$ of $\gamma_1, \ldots, \gamma_n$, $\beta\alpha^{-1}$ are -1.

Note that if $n = 2$, the 2nd line of (b) does NOT appear.

Reflection subgroup

Recall that M_0 is a reflection. ($\Leftrightarrow \mathrm{rank}(M_0 - E_{2^n}) = 1$).
Let $\mathbf{Ref} \triangleleft \mathbf{Mon}$ be the smallest normal subgroup of $\mathbf{Mon}$, which includes the reflection M_0. $\mathbf{Ref}$ is called the reflection subgroup.

By using $\mathbf{Ref}$, we can investigate $\mathbf{Mon}$.
(This idea is similar to [Kato (1997)].)

The reflection subgroup was introduced in [Beukers-Heckman (1989)], to study the monodromy group for ${}_nF_{n-1}$.

Lemma 3 ([G.-Koike (to appear)])

$$\mathbf{Ref} \text{ is finite } \Longleftrightarrow \mathbf{Mon} \text{ is finite}$$

Thus, the finiteness of $\mathbf{Mon}$ is reduced into that of $\mathbf{Ref}$.

Lemma 4

If at least two of $\gamma_1, \ldots, \gamma_n$ are -1, then $\mathbf{Ref} \curvearrowright \mathbb{C}^{2^n}$ is reducible.
For example, if $\gamma_{n-1} = \gamma_n = -1$, then we have a decomposition

$$\begin{aligned} &\mathbf{Ref}^{(n)}(\alpha, \beta, (\gamma_1, \ldots, \gamma_{n-2}, -1, -1)) \\ &\simeq \left(\mathbf{Ref}^{(n-1)}(\alpha, \beta, (\gamma_1, \ldots, \gamma_{n-2}, -1))\right)^2 . \end{aligned}$$

Lemma 5

If at least one of $\gamma_1, \ldots, \gamma_n$ is -1 and $\alpha\beta^{-1}$ is -1, then
$\mathbf{Ref} \curvearrowright \mathbb{C}^{2^n}$ is reducible.
For example, if $\gamma_n = \beta\alpha^{-1} = -1$, then we have a decomposition

$$\begin{aligned} &\mathbf{Ref}^{(n)}(\alpha, \beta, (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, -1)) \\ &\simeq \left(\mathbf{Ref}^{(n-1)}(\alpha, \beta, (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}))\right)^2 . \end{aligned}$$

$(1),(2) \Longrightarrow$ finite irreducible

Irreducibility

It is shown by a straightforward calculation.

Finiteness

By the lemmas, if two of $\gamma_1, \ldots, \gamma_n$, $\beta\alpha^{-1}$ are -1, then $\mathbf{Ref}^{(n)}$ is decomposed into $\mathbf{Ref}^{(n-1)} \times \mathbf{Ref}^{(n-1)}$.
Applying this discussion repeatedly, we obtain

$$\mathbf{Ref}^{(n)} \simeq (\mathbf{Ref}^{(1)})^{2^{n-1}} \quad \text{or} \quad \mathbf{Ref}^{(n)} \simeq (\mathbf{Ref}^{(2)})^{2^{n-2}}.$$

We can check the finiteness of $\mathbf{Ref}^{(1)}$ and $\mathbf{Ref}^{(2)}$
by (1) and the results of [Kato (1997)], respectively. $\square$

finite irreducible $\Longrightarrow$ (1), (2)

Lemma 6

$\mathbf{Mon}^{(n)}(\alpha, \beta, (\gamma_1, \ldots, \gamma_n))$ has a subgroup that is isomorphic to $\mathbf{Mon}^{(k)}(\alpha, \beta, (\gamma_{i_1}, \ldots, \gamma_{i_k}))$ $(k = 1, ..., n-1,\ \{i_1, ..., i_k\} \subset \{1, ..., n\})$.

This lemma shows that "$\mathbf{Mon}^{(n)}$: finite $\Longrightarrow$ (1)".

Lemma 7

Let $n \geq 3$ and $\mathbf{Mon}^{(n)}(\alpha, \beta, \gamma)$ be finite irreducible.
For distinct $i, j, k \in \{1, \ldots, n\}$, $\mathbf{Mon}^{(2)}(\alpha, \alpha\gamma_k^{-1}, (\gamma_i, \gamma_j))$ and $\mathbf{Mon}^{(2)}(\beta, \beta\gamma_k^{-1}, (\gamma_i, \gamma_j))$ are also finite irreducible.

[Sketch of Proof] This follows from

$$x_n^{-a} f\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n}\right) \text{ is a sol. to } E_C(a, b, c)$$

$$\Longleftrightarrow f(\xi_1, \ldots, \xi_n) \text{ is a sol. to } E_C(a, a - c_n + 1, (c_1, \ldots, c_{n-1}, a - b + 1))$$

and Lemma 6. $\square$

By using Lemma 7, we can show that at least $n-2$ of $\gamma_1, \ldots, \gamma_n$ are -1. We may assume $\gamma_3 = \cdots = \gamma_n = -1$.
By using the finiteness conditions of

$$\begin{aligned} &\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, \gamma_2)), \\ &\mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_1, -1)), \quad \mathbf{Mon}^{(2)}(\alpha, \beta, (\gamma_2, -1)), \end{aligned}$$

we can show that at least two of γ_1, γ_2, $\beta\alpha^{-1}$, $\delta_0^{(n)}$ are -1.
Thus, we obtain (2). $\square$

Remark

By the proof, we can see that
if $\mathbf{Mon}^{(n)}$ is finite irreducible, then $\mathbf{Ref}^{(n)}$ is decomposed into a direct product of $\mathbf{Ref}^{(1)}$'s or $\mathbf{Ref}^{(2)}$'s.
The structures of them are studied in [Kato (1997)] and [Kato-Sekiguchi (2010)].

Proof of Theorem 1

Theorem 1

$\pi_1(X, \dot{x})$ is presented as follows.

$\boxed{\text{Generators}}$ $n+1$ loops:

ρ_0 : the loop turning the divisor $S^{(n)} = (R(x) = 0)$ around the point $\left(\frac{1}{n^2}, \ldots, \frac{1}{n^2}\right)$,

ρ_k : the loop turning the divisor $(x_k = 0) \quad (1 \leq k \leq n)$.

$\boxed{\text{Relations}}$

(I) $[\rho_i, \rho_j] = 1$ for $1 \leq i, j \leq n$;

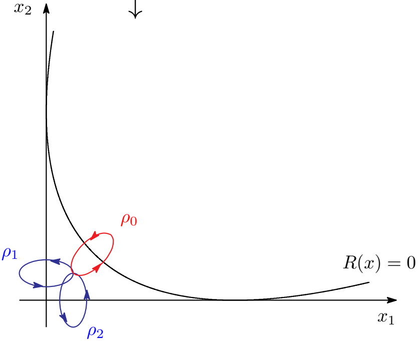
(II) $(\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2$ for $1 \leq k \leq n$;

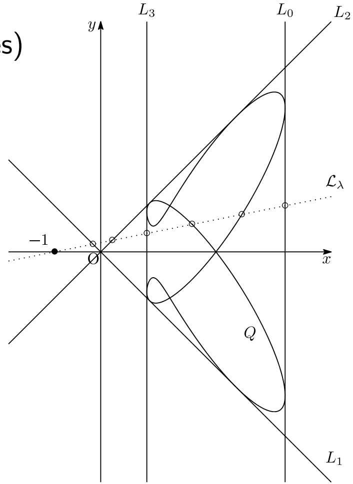
(III) $[(\rho_{i_1} \cdots \rho_{i_p})\rho_0(\rho_{i_1} \cdots \rho_{i_p})^{-1}, (\rho_{j_1} \cdots \rho_{j_q})\rho_0(\rho_{j_1} \cdots \rho_{j_q})^{-1}] = 1$ for $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\} \subset \{1, \ldots, n\}$ with $p, q \geq 1$, $p + q \leq n - 1$ and $I \cap J = \emptyset$.

$n = 2$ & $n = 3$

$n = 3$: $X \cap {}^{\exists}(\text{plane}) \longrightarrow$
(by taking suitable coordinates)

$n = 2$
$\downarrow$





For $n = 2$ ([Kaneko (1981)]) and $n = 3$ ([G.-Kaneko (2018)]), we compute $\pi_1(X)$ by using the theorem of van Kampen-Zariski.

Theorem 1 for general n is shown in [Terasoma; arXiv:1803.06609] (by a more elegant method).

We consider a branched 2^n-covering

$$\phi : \mathbb{C}^n \to \mathbb{C}^n; \quad (\xi_1, \dots, \xi_n) \mapsto (x_1, \dots, x_n) = (\xi_1^2, \dots, \xi_n^2)$$

of $\mathbb{C}^n$. Since

$$R(\phi(\xi)) = \prod_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \Big(1 + \sum_{k=1}^{n} \varepsilon_k \xi_k\Big),$$

$\tilde{X} = \phi^{-1}(X)$ is the complement of hyperplanes in $\mathbb{C}^n$:

$$\begin{aligned} \tilde{X} &= \phi^{-1}(X) \\ &= \mathbb{C}^n - \left(\bigcup_{k=1}^{n} (\xi_k = 0) \cup \bigcup_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \Big(1 + \sum_{k=1}^{n} \varepsilon_k \xi_k = 0\Big) \right). \end{aligned}$$

The restriction $\phi : \tilde{X} \to X$ is a $(\mathbb{Z}/2\mathbb{Z})^n$-Galois covering.

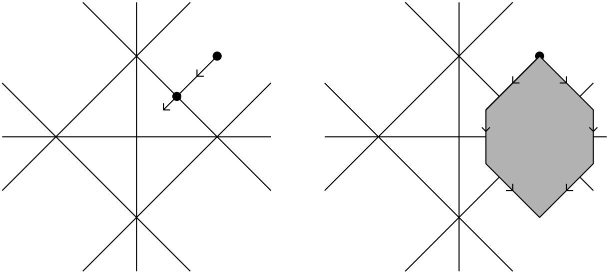
$\phi : \tilde{X} \to X$: a $(\mathbb{Z}/2\mathbb{Z})^n$-Galois covering.
$\tilde{X}$: the complement of real hyperplanes in $\mathbb{C}^n$

- We can take a 2-skeleton $\tilde{X}_2$ of Salvetti complex whose cells are "compatible" with $(\mathbb{Z}/2\mathbb{Z})^n$-action.
- $\pi_1(X) \simeq \pi_1\big(\tilde{X}_2/(\mathbb{Z}/2\mathbb{Z})^n\big)$.
- We write down all relations in $\pi_1\big(\tilde{X}_2/(\mathbb{Z}/2\mathbb{Z})^n\big)$, by using the 2-cells.

$\longrightarrow$ Theorem 1 is proved.

Salvetti complex

- a codim 0 chamber $\longleftrightarrow$ a 0-cell.
- Choose a codim 1 chamber and a codim 0 one.
  $\to$ We obtain a 1-cell τ in $\tilde{X}$. (If we take another codim 0 chamber, then we obtain another 1-cell $\overline{\tau}$.)
- Choose a codim 2 chamber and a codim 0 one.
  $\to$ We obtain a 2-cell σ in $\tilde{X}$.
- Consider the quotient of σ by $(\mathbb{Z}/2\mathbb{Z})^n$-action.
  $\to$ All relations come from the quotients of such 2-cells.



$\boxed{\text{(easy) example}}$ $n = 2$

We find three 2-cells. (It is sufficient to consider $\bullet \in \mathbb{R}^n_{>0}$.)
We can see $a = \rho_2$ and $\bar{b}b = \rho_0$ (in the quotient space).

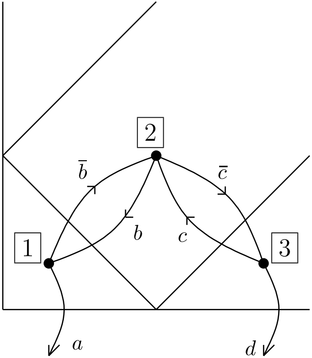
$\boxed{1}$ $a\bar{b}\bar{c} = \bar{b}\bar{c}d$

$\boxed{2}$ $ba\bar{b} = \bar{c}dc$

$\boxed{3}$ $dcb = cba$

By these relations, we have

$$
\begin{aligned}
(\rho_0\rho_2)^2 &= \bar{b}ba\bar{b}ba = \bar{b}\bar{c}dcba \\
&= a\bar{b}\bar{c}dcb = a\bar{b}ba\bar{b}b \\
&= (\rho_2\rho_0)^2.
\end{aligned}
$$



Remark

To write all the relations systematically, Terasoma contracts the spanning complex (= a simply connected subcomplex) to a point.

Summary

- By using explicit matrix presentations, we can obtain a condition when the monodromy group for E_C is finite irreducible.
- The presentation of the fundamental group of the complement of the singular locus of E_C is obtained.

Thank you for your kind attention!

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