Algebraic functions in terms of generalized hypergeometric functions

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Outline

Algebraic functions and the Newton-Puiseux algorithm

Definition

A function $y(x)$ is said to be an algebraic function if it satisfies $f(x, y(x)) = 0$, where $f(x, y) \in \mathbb{C}[x, y]$ is a polynomial in x and y.

Example

$f(x, y) = x^p - y^q = 0$ has the solution $y = x^{\frac{p}{q}}$. In fact, "q" independent solutions to the algebraic equation $f(x, y) = x^p - y^q = 0$, are given by

$$y_k(x) = e^{2\pi i \frac{p}{q} k} x^{\frac{p}{q}} \quad \mathit{for} \quad k = 0, \ldots, q - 1.$$

where we assume p and q are coprime.

Let $f(x,y) = \sum a_{\alpha\beta} x^{\alpha} y^{\beta}$ be a convergent power series and without loss of generality, assume that f is y-general (say of order $m > 0$, *i.e* $a_{0m} \neq 0$ and $a_{0i} = 0$ for $i < m$). Define the carrier of f as

$$\Delta(f) := \{(\alpha, \beta) \in \mathbb{N}^2 | \quad a_{\alpha\beta} \neq 0\}.$$

We now want to find the line which we begin the process. It is the steepest possible line through the lowest point of the carrier on the β-axis.

For each point p of the carrier of f we consider the positive quadrant $p + (\mathbb{R}^+)^2$ moved up to p. From the union of all these displaced with quadrants we construct the convex hull

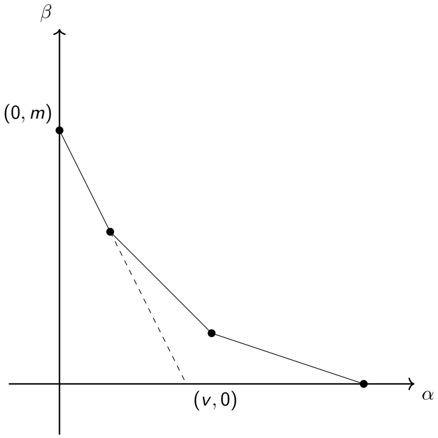
$$conv(\cup_{p \in \Delta(f)}(p + (\mathbb{R}^+)^2))$$

The boundary consists of a compact polygonal path (where all the segments have negative slope) and two half lines. This compact polygonal path is called the Newton polygon of f.

Consider $f(x,y)$ given by

$$f(x,y) = \sum_{\alpha+\mu_0\beta=v} a_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\mu_0\beta>v} a_{\alpha\beta}x^{\alpha}y^{\beta} = \tilde{f}(x,y) + h(x,y) \tag{1.1}$$

Such an $f(x,y)$ has the following Newton polygon:



The line(dotted) equation is : $\alpha + \mu_0 \beta = v$.

v is the intercept on the α-axis of the line through $(0, m)$ with slope $-\frac{1}{\mu_0}$.
For the first approximate solution, put $y = cx^{\mu_0}$ in $\tilde{f}(x, y) = 0$.

$$\begin{aligned}\tilde{f}(x, cx^{\mu_0}) &= \sum_{\alpha+\mu_0\beta=v} a_{\alpha\beta}x^{\alpha}c^{\beta}x^{\mu_0\beta} \\ &= x^v \sum_{\alpha+\mu_0\beta=v} a_{\alpha\beta}c^{\beta} = x^v g(c) = 0\end{aligned}$$

where g is a polynomial of degree m.

As $a_{\alpha\beta} \neq 0$, $g(c)$ has a non-zero root c_0.
Thus, $y_0 = c_0 x^{\mu_0}$ is a solution to the equation $\tilde{f}(x, y) = 0$.
Since $\mu_0 \in \mathbb{Q}_{>0}$, $\mu_0 = \frac{p_0}{q_0}$ with $gcd(p_0, q_0) = 1$.
Set $x_1 := x^{\frac{1}{q_0}}$ and to improve the approximate solution put $y = x_1^{p_0}(c_0 + y_1)$ in throughout $f(x, y)$.

This gives us a new power series $f(x_1^{q_0}, x_1^{p_0}(c_0 + y_1))$.
$x_1^{vq_0}$ divides this power series by (1.1)

$$f(x_1^{q_0}, x_1^{p_0}(c_0 + y_1)) = x_1^{vq_0} f_1(x_1, y_1).$$

$f_1(x_1, y_1)$: y_1-general of order $m_1 \leq m$.

Again, construct the Newton polygon of f_1. Let $-\frac{1}{\mu_1}$ be its steepest negative slope, then $\mu_1 = \frac{p_1}{q_1}$ and obtain an approximate solution $y = c_1 x^{\mu_1}$ such that $g_1(c_1) = 0$.

Set $x_2 := x_1^{\frac{1}{q_1}}$, and improve the solution by putting $y_1 = x_2^{p_1}(c_1 + y_2)$ throughout $f_1(x_1, y_1) = 0$.

Thus,

$$f_1(x_2^{q_1}, x_2^{p_1}(c_1 + y_2)) = x_2^{v_1 q_1} f_2(x_2, y_2)$$

$f_2 : y_2-$general of order $m_2 \leq m_1$.
Continuing this process eventually yields a sequence of approximate solutions

$$\begin{aligned} y &= x^{\mu_0}(c_0 + x_1^{\mu_1}(c_1 + x_2^{\mu_2}(c_2 + \cdots))) \\ &= c_0 x^{\mu_0} + c_1 x^{\mu_0} x_1^{\mu_1} + c_2 x^{\mu_0} x_1^{\mu_1} x_2^{\mu_2} + \cdots \end{aligned}$$

The process breaks off when the Newton polygon of f_j is a single point for some j.

Trinomial algebraic function

Consider a general trinomial equation given by

$$\varphi(x,y) = y^n + xy^p - 1 = 0, \quad (n > p). \tag{2.1}$$

Remark

The singular points(discriminant) of the equation (2.1) *are given by*

$$x^n = (-1)^p \frac{n^n}{p^p q^q}, \quad (q = n - p).$$

If $y(x)$ is a solution of (2.1), then

$$y_k(x) = \epsilon_k y((\epsilon_k)^p x)$$

is also a solution to $\varphi(x,y) = y^n + xy^p - 1 = 0$, where $\epsilon_k = e^{\frac{2\pi ik}{n}}$ for $k = 0, 1, \ldots n-1$.

Theorem

(Mellin Inversion Theorem) Let $z = \sigma + it \in \mathbb{C}$.

(i) Suppose $f(z) \in \mathcal{O}(\{\sigma : \alpha < \sigma < \beta\})$.

(ii) Assume $\lim_{|t|\to\infty} f(z)$ *converges uniformly in every strip* $\alpha + \delta \leq |\sigma| \leq \beta - \delta$, *where* $\delta > 0$.

Then for a real positive x define

$$g(x) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} x^{-z} f(z) dz$$

and

$$f(z) = \int_0^\infty x^{z-1} g(x) dx.$$

Lemma

For a solution $y(x)$ of (2.1),the Mellin transformation of $y(x)$ reads

$$\int_0^\infty y(x)x^{z-1}dx = \frac{1}{n}\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)}.$$

Example

For $n = 2, p = 1$,

$$\int_0^\infty \left(\frac{-x + \sqrt{x^2 + 4}}{2} \right) x^{z-1} dx = \frac{1}{2} \frac{\Gamma(z)\Gamma(\frac{1-z}{2})}{\Gamma(\frac{1+z}{2} + 1)}.$$

Example

For $n = 3, p = 1$,

$$\int_0^\infty \left(\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{x}{3}\right)^3}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} + \left(\frac{x}{3}\right)^3}} \right) x^{z-1} dx$$

$$= \frac{1}{3} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{1+2z}{3} + 1)}.$$

By Mellin inversion theorem,

Proposition

The Mellin inversion formula gives a special solution of $\varphi(x,y) = y^n + xy^p - 1 = 0.$

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{n}\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z}\,dz, \quad 0 < c < \frac{1}{p}. \qquad (2.2)$$

Proof.

Apply Stirling's formula for the gamma function to the above integrand for $x = |x|e^{-i\theta}$.

$$\frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)}x^{-z} \sim \sqrt{2\pi}e^{-|t|\frac{\pi p}{n}}e^{-\theta t}.$$

The above integral (2.2) converges in the angular domain

$$-\frac{\pi p}{n} \leq \theta \leq \frac{\pi p}{n}$$

as $|t| \to \infty$, where $z = s + it \in \mathbb{C}$. Hence satisfies the conditions in theorem (2.1). $\square$

Consider

$$y(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{n} \frac{\Gamma(z)\Gamma(\frac{1-pz}{n})}{\Gamma(\frac{1+qz}{n}+1)} x^{-z} dz,$$

Now take residues at $z = -k$, where $k \in \mathbb{Z}_{\geq 0}$ of the above integral.

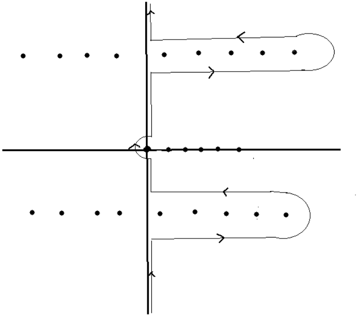


Figure 2.1: Barnes Contour

Eventually, we obtain

$$Y_0(X) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1+pk}{n})(-1)^{\frac{pk+1}{n}+k}}{\Gamma(\frac{1-qk}{n}+1)k!} (X^{\frac{1}{n}})^{Nk+r} \tag{2.3}$$

Also, put

$$Y_t(X) := Y_0(\epsilon^t X^{\frac{1}{n}}),$$

where $N = n - qr < 0$ and $\epsilon^t = e^{\frac{2\pi it}{n}}$, for $t = 0, 1, \cdots, n-1$.

Example

Let us try to find a solution of the algebraic equation $\varphi(x,y) = y^3 + xy - 1 = 0$, by using Mellin Inversion formula

$$y(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz.$$

By using the relation $\Gamma(z+3) = (z+2)(z+1)z\Gamma(z)$ we obtain

$$\left[x^3\left(\frac{1-2\vartheta_x}{3}\right)\left(\frac{-2-2\vartheta_x}{3}\right)\left(\frac{1+\vartheta_x}{3}\right) + \vartheta_x(\vartheta_x - 2)(\vartheta_x - 1)\right] y(x) = 0. \tag{2.4}$$

$\vartheta_x = x\frac{d}{dx}$.

Applying the change of variables $x^3 = t$ and then $-\frac{2^2}{3^3}t = s$ eventually yields the hypergeometric differential equation

$$\left[-s(\vartheta_s - \frac{1}{6})(\vartheta_s + \frac{1}{3})^2 + (\vartheta_s)(\vartheta_s - \frac{2}{3})(\vartheta_s - \frac{1}{3})\right]\psi(s) = 0 \quad (2.5)$$

where $y((-\frac{3^3}{2^2}s)^{\frac{1}{3}}) = \psi(s)$. In fact $\psi(s)$ is a Pochhammer hypergeometric function of the form

$$\psi(s) = {}_3F_2(-\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}; s) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{6})_k(\frac{1}{3})_k^2}{(-\frac{2}{3})_k(-\frac{1}{3})_k k!} s^k.$$

$$y(x) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{6})_k(\frac{1}{3})_k^2}{(-\frac{2}{3})_k(-\frac{1}{3})_k k!} \left(-\frac{2^2}{3^3}x^3\right)^k$$

$$= {}_3F_2(-\frac{1}{6}, \frac{1}{3}, \frac{1}{3}; -\frac{2}{3}, -\frac{1}{3} | \left(-\frac{2^2}{3^3}x^3\right))$$

$$= \sum_{k=0}^{\infty} \frac{(-\frac{1}{6})_k(\frac{1}{3})_k(-\frac{2}{3}+k)}{(-\frac{1}{3})_k(-\frac{2}{3})k!} \left(-\frac{2^2}{3^3}x^3\right)^k .$$

The equation (2.5) has a non-algebraic solution near $\frac{1}{s} = 0$.

$$\left(\frac{1}{s^{\frac{1}{3}}}\right) \log\left(\frac{1}{s}\right) + o\left(\frac{1}{s^{\frac{1}{3}}}\right).$$

Proposition

In general, for a given algebraic equation of the form $\varphi(x,y) = y^n + xy^p - 1 = 0$, the general hypergeometric differential equation satisfied by $y(x)$ can be given by

$$\left[\prod_{l=0}^{n-1}(\vartheta_x - l) - \frac{(-p)^p q^q}{n^n} x^n \prod_{j=0}^{p-1}(\vartheta_x + \frac{1+nj}{p}) \prod_{k=0}^{q-1}(\vartheta_x + \frac{-1+nk}{q})\right] y(x) =$$

and after the change of variables $s = \frac{(-p)^p q^q}{n^n} x^n$, and $\vartheta_x = n\vartheta_s$ one gets

$$\left[\prod_{l=0}^{n-1}(\vartheta_s - \frac{l}{n}) - s\prod_{j=0}^{p-1}(\vartheta_s + \frac{1}{np} + \frac{j}{p}) \prod_{k=0}^{q-1}(\vartheta_s - \frac{1}{nq} + \frac{k}{q})\right] Y(s) = 0.$$

Thus,

$$Y(s) = {}_nF_{n-1}\left(\overrightarrow{\alpha}; \overrightarrow{\beta}|s\right) \tag{2.6}$$

where $s = \frac{(-p)^p q^q}{n^n} x^n$ and $Y(s) = y(s^{\frac{1}{n}})$.

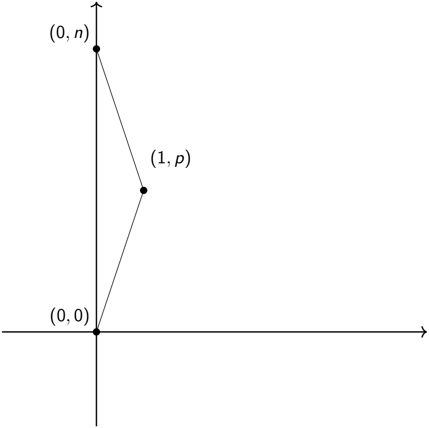
$$\overrightarrow{\alpha} = \left(\frac{1}{np}, \frac{1}{np} + \frac{1}{p} \ldots, \frac{1}{np} + \frac{p-1}{p}, -\frac{1}{nq}, -\frac{1}{nq} + \frac{1}{q}, \ldots, -\frac{1}{nq} + \frac{q-1}{q}\right)$$

$$\overrightarrow{\beta} = \left(-\frac{1}{n}, \ldots, -\frac{n-1}{n}\right).$$

Algebraic functions in terms of hypergeometric series

Newton polygon of

$$\varphi(x, y) = y^n + xy^p - 1 = 0 : \tag{3.1}$$



Consider a general trinomial algebraic equation of the following form:

$$f(X,Y) = Y^n - XY^p + X^r = 0,$$

with $(n-p) > \frac{p}{r-1}$ and $gcd(p, r-1) = 1$.

Let us try to express the solutions of the above general trinomial algebraic equation $f(X,Y)$ in terms of hypergeometric series.

One can convert the equation $\varphi(x, y) = 0$ into the equation $f(X, Y) = 0$ by the aid of the following new variables

$$Y(X) = (-1)^{\frac{1}{n}} X^{\frac{r}{n}} y(x) \tag{3.2}$$

and

$$x = (-1)^{\frac{p}{n}} X^{\frac{n-qr}{n}} \tag{3.3}$$

Recall

$$[\prod_{l=0}^{n-1}(\vartheta_x - l) - \frac{(-p)^p q^q}{n^n} x^n \prod_{j=0}^{p-1}(\vartheta_x + \frac{1+nj}{p}) \prod_{k=0}^{q-1}(\vartheta_x + \frac{-1+nk}{q})]y(x) = 0.$$

By using (3.2) and (3.3) one obtains

$$[\frac{n^n}{p^p q^q} X^{-N} \prod_{l=0}^{n-1}(\frac{n}{N}\vartheta_X - l) - \prod_{j=0}^{p-1}(\frac{n}{N}\vartheta_X + \frac{1+nj}{p}) \prod_{k=0}^{q-1}(\frac{n}{N}\vartheta_X + \frac{-1+nk}{q})]y((-1)^{\frac{p}{n}} X^{\frac{N}{n}}) = 0 \tag{3.4}$$

where $N = n - qr < 0$, and $q = n - p$.

In making use of the change of variable $\frac{n^n}{p^p q^q} X^{-N} = s$, hence $\vartheta_X = -N\vartheta_s$, one gets

$$\left[s \prod_{l=0}^{n-1} (\vartheta_s + \frac{l}{n}) - \prod_{j=0}^{p-1} (\vartheta_s - \frac{1}{np} - \frac{j}{p}) \prod_{k=0}^{q-1} (\vartheta_s + \frac{1}{nq} - \frac{k}{q}) \right] \psi(s) = 0. \tag{3.5}$$

Proposition

In general, solutions of the algebraic equation

$$f(X,Y) = Y^n - XY^p + X^r = 0, \tag{3.6}$$

with $(n-p) > \frac{p}{r-1}$ *and* $\gcd(p, r-1) = 1$, *are of the form*

$$\tilde{Y}_t(X) = \tilde{Y}(\epsilon^t X^{\frac{1}{p}}) \quad \epsilon^t = e^{\frac{2\pi i t}{p}} \quad \textit{for} \quad t = 0, 1, \ldots, p-1.$$

and

$$-\tilde{\tilde{Y}}_s(X) = \tilde{\tilde{Y}}(\omega^s X^{\frac{1}{q}}) \quad \omega^s = e^{\frac{2\pi i s}{q}} \quad \textit{for} \quad s = 0, 1, \ldots, q-1.$$

where, $\tilde{Y}_t$ *and* $\tilde{\tilde{Y}}_s$ *are the solutions of the hypergeometric differential equation* (3.5)*, derived from the Mellin inversion formula.*

Remark

Remark on (3.4),

(i) *If* $\gcd(n, p) = d > 1$, *then solutions to the differential equation* (3.4) *are all algebraic functions with associated equation* $y^n + xy^p - 1 = 0$.

(ii) *If* $d = 1$ *and* $p < n - 1$, *then there exists 1-dimensional logarithmic solution space of* (3.4) *over* $\mathbb{C}$, *spanned by* $\sum_{k=0}^{n-1} Y_k \log(Y_k)$, *where* $f(X, Y_t(X)) = 0$ (3.6) *for* $t = 0, 1, \cdots, n-1$.

(iii) *If* $d = 1$ *and* $p = n - 1$, *then the set* $\{Y_0, \ldots, Y_{n-2}, X\}$ *gives* $n-$*dimensional solution space over* $\mathbb{C}$ *to the differential equation* (3.4).

If $p < n-1$,

$$\sum_{t=0}^{p-1} \tilde{Y}_t(X) - \sum_{s=0}^{q-1} \tilde{\tilde{Y}}_s(X) = 0$$

These algebraic solutions span a $(n-1)-$dimensional solution subspace to (3.4) over $\mathbb{C}$.
If $p = n-1$, then

$$\sum_{t=0}^{p-1} \tilde{Y}_t(X) - \sum_{s=0}^{q-1} \tilde{\tilde{Y}}_s(X) = X.$$

Example

Example

Assume that we are given the algebraic equation

$$f(X,Y) = Y^3 - XY + X^4 = 0.$$

For a solution $y(x)$ of the algebraic equation $\varphi(x,y) = y^3 + xy - 1 = 0$, let

$$y_0(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z} dz,$$

Also, employ Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

Claim: There exists a periodic function $\phi(z)$ such that

$$\frac{\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})}\phi(z) = \frac{\Gamma(\frac{-1-2z}{3})}{\Gamma(\frac{2+z}{3})}(-1)^{-z},$$

with $\phi(z+3) = \phi(z)$.

Hence, one can obtain

$$\tilde{y}_0(x) = \frac{1}{3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(\frac{-1-2z}{3})}{\Gamma(\frac{2+z}{3})}(-x)^{-z}dz$$

which is also a solution to the hypergeometric differential equation (2.4)

$$\left[x^3\left(\frac{1-2\vartheta_x}{3}\right)\left(\frac{-2-2\vartheta_x}{3}\right)\left(\frac{1+\vartheta_x}{3}\right)+\vartheta_x(\vartheta_x-2)(\vartheta_x-1)\right]y(x)=0$$

Calculating the residue yields the following series

$$y_0(x) = -\frac{1}{3}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=3k+1} \frac{\Gamma(z)\Gamma(\frac{1-z}{3})}{\Gamma(\frac{4+2z}{3})} x^{-z}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)(-1)^k}{\Gamma(2k+2)k!} x^{-3k-1}. \tag{3.7}$$

Since

$$\operatorname*{Res}_{z=3k+1} \Gamma(\frac{1-z}{3})dz = \operatorname*{Res}_{s=-k} \Gamma(s)(-3)ds$$

Also

$$\tilde{y}_0(x) = -\frac{1}{3}\sum_{k=0}^{\infty} \operatorname*{Res}_{z=\frac{3k-1}{2}} \frac{\Gamma(z)\Gamma(\frac{-1-2z}{3})}{\Gamma(\frac{2+z}{3})}(-x)^{-z}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^k}{\Gamma(\frac{k+1}{2})k!}(-x)^{\frac{1-3k}{2}}. \tag{3.8}$$

Since

$$\operatorname*{Res}_{z=\frac{3k-1}{2}} \Gamma(\frac{-1-2z}{3})dz = \operatorname*{Res}_{s=-k} \Gamma(s)\left(-\frac{3}{2}\right) ds$$

Now, using

$$Y(X) = (-1)^{\frac{1}{3}} X^{\frac{4}{3}} y(x) \tag{3.9}$$

and

$$x = (-1)^{\frac{1}{3}} X^{-\frac{5}{3}} \tag{3.10}$$

in $y_0(x)$ and $\tilde{y_0}(x)$, we obtain

$$\begin{aligned} \tilde{Y_0}(X) &= (-1)^{\frac{1}{3}} X^{\frac{4}{3}} \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)(-1)^k}{\Gamma(2k+2)k!} ((-1)^{\frac{1}{3}} X^{-\frac{5}{3}})^{-3k-1} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(3k+1)}{\Gamma(2k+2)k!} X^{5k+3} \end{aligned} \tag{3.11}$$

and

$$\tilde{\tilde{Y}}_0(X) = (-1)^{\frac{1}{3}} X^{\frac{4}{3}} \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^k}{\Gamma(\frac{k+1}{2})k!} (-(-1)^{\frac{1}{3}} X^{-\frac{5}{3}})^{\frac{1-3k}{2}}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}} \tag{3.12}$$

,

$$\tilde{\tilde{Y}}_1(X) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!} (e^{\pi i} X^{\frac{1}{2}})^{5k+1}$$

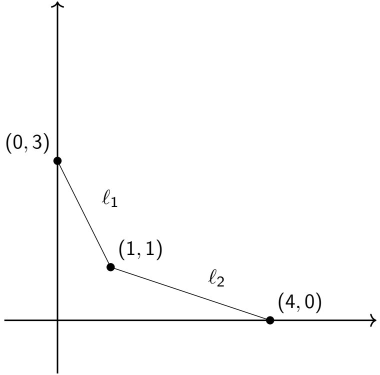
$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3k-1}{2})}{\Gamma(\frac{k+1}{2})k!} X^{\frac{5k+1}{2}} \tag{3.13}$$

.

On the other hand, we apply Newton's algorithm to the algebraic equation

$$f(X, Y) = Y^3 - XY + X^4 = 0.$$

Newton polygon of $f(X, Y)$:



On the line ℓ_1:
$\mu_0 = \frac{1}{2}$, so we start with putting $Y_0 = c_0 X^{\frac{1}{2}}$ into the equation $\tilde{f}(X, Y) = Y^3 - XY = 0$, and get

$$c_0^3 X^{\frac{3}{2}} - c_0 X^{\frac{3}{2}} = c_0(c_0^2 - 1) X^{\frac{3}{2}} = 0,$$

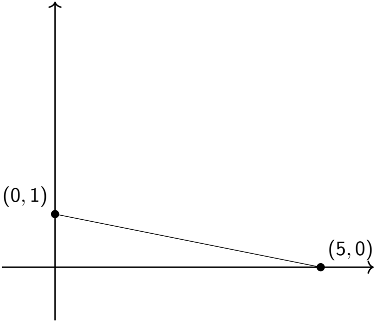
which implies $c_0 = \pm 1$. Choose the branch $c_0 = 1$, then $Y_0 = X^{\frac{1}{2}}$. We let $X^{\frac{1}{2}} = X_1$, and compute the second approximation

$$\begin{aligned} f(X_1^2, X_1(1+Y_1)) &= X_1^3(1+Y_1)^3 - X_1^3(1+Y_1) + X_1^8 \\ &= X_1^3 f_1(X_1, Y_1) \end{aligned}$$

where

$$f_1(X_1, Y_1) = Y_1^3 + 3Y_1^2 + 2Y_1 + X_1^5$$

Newton polygon of $f_1(X_1, Y_1)$:



$$2Y_1 + X_1^5 = 0 \quad \Rightarrow \quad Y_1 = -\frac{X_1^5}{2}, \quad c_1 = -\frac{1}{2}$$

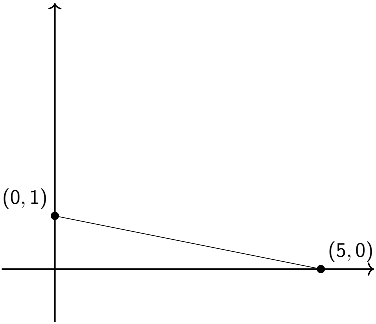
For the third approximation put $f_1(X_1, X_1^5(-\frac{1}{2}+Y_2))$

$$f_1(X_1, X_1^5\left(-\frac{1}{2}+Y_2\right)) = X_1^5 f_2(X_1, Y_2)$$

where

$$f_2(X_1, Y_2) = X_1^{10}\left(-\frac{1}{2}+Y_2\right)^3 + 3X_1^5\left(-\frac{1}{2}+Y_2\right)^2 + 2Y_2.$$

Newton polygon of f_2:



$$2Y_2 + \frac{3}{4}X_1^5 = 0 \quad \Rightarrow \quad Y_2 = -\frac{3}{8}X_1^5, \quad c_2 = -\frac{3}{8}$$

Lastly, we apply the algorithm one step further that is, we put

$$f_2(X_1, X_1^5(-\frac{3}{8} + Y_3)) = X_1^{10}[2Y_3 - 3X_1^5(-\frac{3}{8} + Y_3) + higher \quad order \quad ter$$
$$= X_1^{10} f_3(X_1, Y_3)$$

where

$$f_3(X_1, Y_3) = 2Y_3 + \frac{9}{8}X_1^5 - \frac{1}{8}X_1^5 + higher \quad order \quad terms$$

similarly

$$2Y_3 + X_1^5 = 0 \quad \Rightarrow \quad Y_3 = -\frac{1}{2}X_1^5, \quad c_3 = -\frac{1}{2}.$$

We obtain a sequence of approxiamte solutions as

$$
\begin{aligned}
Y(X) &= X_1(1 + X_1^5(-\frac{1}{2} + X_1^5(-\frac{3}{8} + -\frac{1}{2}X_1^5 + \cdots)))\\
&= X^{\frac{1}{2}} - \frac{1}{2}X^3 - \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots
\end{aligned}
$$

$$\tilde{\tilde{Y}}_{psx}^{(1)}(X) = X^{\frac{1}{2}} - \frac{1}{2}X^3 - \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots$$

There is another branch on the line ℓ_1. Namely, $\mu_0 = \frac{1}{2}$ and $c_0 = -1$.

$$\tilde{\tilde{Y}}_{psx}^{(0)}(X) = -X^{\frac{1}{2}} - \frac{1}{2}X^3 + \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots$$

For the third branch we look at the line $\ell_2(\mu_0 = 3,\ c_0 = 1)$.

$$\tilde{Y}_{psx}^{(0)}(X) = X^3 + X^8 + 3X^{13} + 12X^{18} + \cdots$$

$$-\tilde{\tilde{Y}}_1(X) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3k-1}{2})}{\Gamma(\frac{k+1}{2})k!}X^{\frac{5k+1}{2}} = X^{\frac{1}{2}} - \frac{1}{2}X^3 - \frac{3}{8}X^{\frac{11}{2}} - \frac{1}{2}X^8 + \cdots$$
$$= \tilde{\tilde{Y}}_{psx}^{(1)}(X),$$

$$-\tilde{\tilde{Y}}_0(X) = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3k-1}{2})(-1)^{1-k}}{\Gamma(\frac{k+1}{2})k!}X^{\frac{5k+1}{2}} = -X^{\frac{1}{2}} - \frac{1}{2}X^3 + \frac{3}{8}X^{\frac{11}{2}}$$
$$-\frac{1}{2}X^8 + \cdots$$
$$= \tilde{\tilde{Y}}_{psx}^{(0)}(X),$$

and

$$\tilde{Y}_0(X) = \sum_{k=0}^{\infty}\frac{\Gamma(3k+1)}{\Gamma(2k+2)k!}X^{5k+3} = X^3 + X^8 + 3X^{13} + \cdots = \tilde{Y}_{psx}^{(0)}(X).$$

Finally, the hypergeometric differential equation satisfied by for every algebraic function satisfying

$$f(X,Y) = Y^3 - XY + X^4 = 0$$

is

$$\left[\frac{3^3}{2^2}X^5\vartheta_X(\vartheta_X+\frac{5}{3})(\vartheta_X+\frac{10}{3}) - (\vartheta_X-\frac{5}{3})(\vartheta_X+\frac{5}{6})(\vartheta_X-\frac{5}{3})\right]\tilde{Y}_0(X) = 0$$

$$\left[\frac{3^3}{2^2}X^5\vartheta_X(\vartheta_X+\frac{5}{3})(\vartheta_X+\frac{10}{3}) - (\vartheta_X-\frac{5}{3})(\vartheta_X+\frac{5}{6})(\vartheta_X-\frac{5}{3})\right]\tilde{\tilde{Y}}_j(X) = 0$$

for $0 \leq j \leq 1.$

We see that

$$\tilde{\tilde{Y}}_{psx}^{(0)}(X) + \tilde{\tilde{Y}}_{psx}^{(1)}(X) + \tilde{Y}_{psx}^{(0)}(X) = 0.$$

$\tilde{\tilde{Y}}_{psx}^{(0)}(X)$, $\tilde{\tilde{Y}}_{psx}^{(1)}$, $\tilde{Y}_{psx}^{(0)}(X)$ span a $2-$dimensional subspace to the above hypergeometric differential equation. There exists a non-algebraic solution to the above differential equation namely,

$$X^{\frac{5}{3}} \log X + o(X^{\frac{5}{3}}).$$

Conclusion

In general when n, p, r are large positive integers, finding the solutions of a given algebraic equation of the form

$$f(X,Y) = Y^n - XY^p + X^r = 0 \quad with \quad (n-p) > \frac{p}{r-1}$$

by means of Newton's algorithm requires much more complicated calculations because of the higher order terms which occur at each approximation step. The hypergeometric series $\tilde{Y}_t$ and $\tilde{\tilde{Y}}_s$ do not require such tough computations. Also, infinitely many coefficients and powers of X are known.

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