On the Topology of Hyperplane Arrangements and some of their Quotients

Michael Lönne

University of Bayreuth

February 18, 2020

subject classification

topology, algebraic geometry,
group theory, singularity theory

Outline

outline

- the braid monodromy revisited
- the Zariski van Kampen theorem
- fundamental groups of discriminant complements

braid group invariant

Definition of BrM, the braid monodromy group

Given a space X of simple monic polynomials p_x of degree k, then

$$BrM_X \quad = \quad p_* \pi_1(X),$$

where the map to the configuration space of $\mathbb{C}$,

$$x \mapsto p_x^{-1}(0) \in \Sigma_k(\mathbb{C})$$

induces the braid monodromy on fundamental groups

$$p_* : \pi_1 X \to \mathbf{B}_k = \pi_1 \Sigma_k(\mathbb{C})$$

Remarks

- simple polynomials are those which have number of zeroes equal to the degree (without multiplicities)
- local problem, i.e. X punctured disc, solvable by Newton-Puiseux.
- global problem: how to fit local solutions together

affine divisor complements

typical situation

- V affine space of polynomials,
- $X \subset V$ Zariski-open subset of simple *monic* polynomials
- $p^{-1}(0) =: \mathcal{D} \subset V \times \mathbb{C}$ a *horizontal* divisor, e.g. a discriminant

hyperplane arrangement situation

With respect to suitable distinguished variable z

p is the product of linear factors monic in z

Theorem (Zariski, van Kampen)

Suppose p monic, braid monodromy group BrM_X generated by $\{\beta_1, ..., \beta_r\} \subset \mathrm{Br}_k$, then $\pi_1(V \times \mathbb{C} - \mathcal{D})$ is finitely presented as

$$\langle t_1, ..., t_k \,|\, t_i^{-1} t_i^{\beta_j}, i \leq k, j \leq r \rangle.$$

Hurwitz action:

$$t_i^{\sigma_j} = \begin{cases} t_{i+1} & \text{if } j = i, \\ t_i t_{i-1} t_i^{-1} & \text{if } j = i-1, \\ t_i & \text{else.} \end{cases}$$

van Kampen

Proof (Idea of)

The locally trivial fibration on $X \times \mathbb{C} - \mathcal{D}$ has a section.
The boundary map $\pi_2 \to \pi_1$ in the long homotopy sequence is trivial.
The section provides a semi-direct product structure:

$$\pi_1(X \times \mathbb{C} - \mathcal{D}) \cong \pi_1(x_0 \times \mathbb{C} - p_x^{-1}(0)) \ltimes \pi_1(X)$$

where the second acts on the former by Hurwitz automorphisms.
This gives a presentation relying on $\pi_1(X) = \langle a_j | \mathcal{R} \rangle$:

$$\pi_1(X \times \mathbb{C} - \mathcal{D}) \cong \langle t_1, ..., t_k, a_1, \ldots, a_{r'} \,|\, a_j^{-1} t_i^{-1} a_j t_i^{p_* a_j}, i \leq k, j \leq r', \mathcal{R} \rangle.$$

surjects onto $\pi_1(V \times \mathbb{C} - \mathcal{D})$ with kernel normally generated by a_j:

$$\pi_1(V \times \mathbb{C} - \mathcal{D}) \cong \langle t_1, ..., t_k \,|\, t_i^{-1} t_i^{p_* a_j}, i \leq k, j \leq r' \rangle.$$

and the claim follows by replacing the generators of BrM_X.

Remark

A careful choice of generators β_j can reduce the number of relations.

Remark

- π_1 is invariant under some modifications of BrM_X
- replace generator $\sigma_1^2\sigma_3^2$ by generators σ_1^2 and σ_3^2
- pretend that distinct singularities are in distinct fibres.

discriminant knot group of some Brieskorn-Pham singularities

new topic

discriminant complements of hypersurface singularities

object of study

A polynomial f on affine space $\mathbb{C}^k$ with an isolated singularity in 0.

Remark

Their discriminant complements may be Eilenberg-MacLane spaces of their fundamental groups.

Remark

With some care, the argument can be used to get knot groups for some A-discriminants.

bifurcation complement

Unfolding spaces in singularity theory:

- $V_f = f + \mathbb{C}[x_1, ..., x_n]_{trunc}$, an unfolding space of

$$f = x_1^{d_1} + x_2^{d_2} + \cdots + x_n^{d_n}, \text{ (Brieskorn-Pham polynomial)}$$

- u_ν, the coefficients of monomials $g_\nu = \prod_i x_i^{\nu_i}$ are coordinates.
- get $P \in \mathbb{C}[u_\nu][z]$, monic in z, by eliminating x_i from

$$F = f(x_1, ..., x_n) - z + \sum u_\nu \prod x_i^{\nu_i} \ = \ \frac{\partial}{\partial x_i} F \ = \ 0.$$

- $X_f = \{u \in V_f \mid P_u \text{ simple}\}$ the bifurcation complement.
- BrM_{X_f} is an invariant of f.

Remark

Due to a genericity argument, unfolding over $\mathbb{C}[x_1, ..., x_n]_{\deg \le 2}$ is sufficient.

Over a generic line in unfolding space, we get a plane curve $\mathcal{C}$, the plane section of $\mathcal{D}$, such that the restricted fibre bundle is given as the complement of the vertical lines through singular points of $\mathcal{C}$. These are of two kinds (as opposed to the uniqueness of a Morse singularity), the ordinary node and the ordinary cusp, corresponding to the two distinct strata in $\mathcal{D}$ of codimension one.

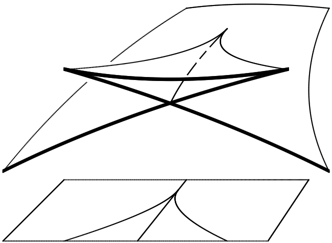


Figure: plane section of discriminant

- suffices to understand BrM for unfolding by linear terms. (technical: versal braid monodromy)

Hefez Lazzeri base

We consider polynomials $f(x, y) = x^3 + y^{\ell+1}$.

The generic function is Morse in the unfolding of f by linear terms

$$F(x, y, a, b) \quad := \quad x^3 - 3ax + y^{\ell+1} - \frac{\ell+1}{\ell} by,$$

$\tilde{f} = F(x, y, 1, 1)$ has critical values $z_i = 2 + y_i, z_{\ell+i} = -2 + y_i$, where the y_i are the ℓ solutions to $y^\ell = 1/\ell$ ordered by increasing argument.

Hefez Lazzeri base

At parameters $a = 1, b = 1$,
$\tilde{f} = F(x, y, 1, 1)$ has critical values $z_i = 2 + y_i, z_{\ell+i} = -2 + y_i$, where the y_i are the ℓ solutions to $y^\ell = 1/\ell$ ordered by increasing argument.

The geometric basis $\{t_i, 1 \leq i \leq 2\ell\}$ for $\tilde{f}$, can be understood from the figure

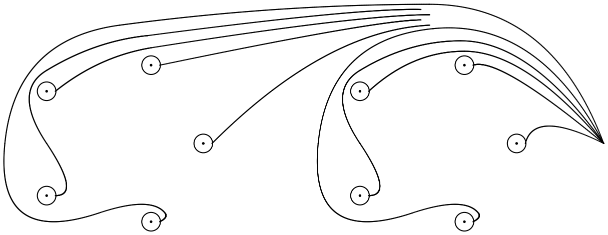


Figure: Hefez Lazzeri system in case $\ell = 5$

hyperplane model

The discriminant has a formal factorisation:

$$p(a,b,z) = \prod_{j=1}^{\ell} \left(z - 2a^{\frac{3}{2}} - \xi^j \left(\frac{b^{\ell+1}}{\ell} \right)^{\frac{1}{\ell}} \right) \left(z + 2a^{\frac{3}{2}} - \xi^j \left(\frac{b^{\ell+1}}{\ell} \right)^{\frac{1}{\ell}} \right),$$

By non-linear change of coordinates $\rightarrow$ hyperplane arrangement

$$q(v,w,z) = ((z-v)^\ell - w^\ell)((z+v)^\ell - w^\ell),$$

$v^2 = 4a^3$ and $(\ell w)^\ell = b^{\ell+1}$

Proposition

The braid monodromy of the line arrangement with constant η

$$((z-v)^\ell - \eta)((z+v)^\ell - \eta)$$

is generated by full twists σ_α on all arcs α such that

$$\alpha = \alpha_{i,j} : 1 \leq i - j < \ell$$

Proposition

The braid monodromy of the plane arrangement

$$((z-v)^\ell - w^\ell)((z+v)^\ell - w^\ell)$$

has in addition the full-twists on the first and the second set of ℓ punctures.

(up to modifications not affecting the fundamental group)

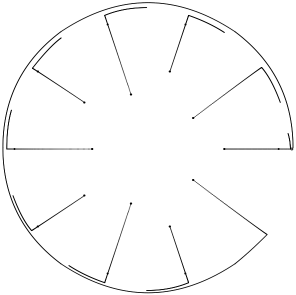
Proposition

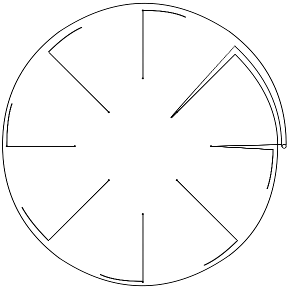
The braid monodromy of the discriminant for the family of functions

$$x^3 - 3ax + y^{\ell+1} - \frac{\ell+1}{\ell} y$$

is generated by

- full twists σ_α^2 on all arcs $\alpha_{i,j}$ with $1 \le i - j < \ell$
- cusp-twists σ_α^3 on all arcs $\alpha_{i,i}$ with $1 \le i \le \ell$.





Fast forward of the remaining steps of the proof

- The versal braid monodromy of the second family.
- A suitable choice of generators for the braid monodromy group.

Theorem

BrM_{X_f} of the polynomial $x^3 + y^{\ell+1}$ is generated by:

- σ_{ij}^2 in case ${}_i\cdot \qquad \cdot_j$
- σ_{ij}^3 in case ${}_i\cdot\!-\!\!\!-\!\cdot_j$,
- $\sigma_{ij}^{\pm 2}\sigma_{ik}^2\sigma_{ij}^{\mp 2}$ in case ${}_i\cdot \underset{j}{\vee} \cdot_k$ ($\pm = \varepsilon_{ijk}$ antisymmetric).

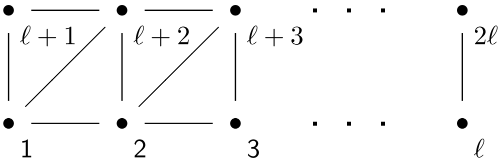


Figure: Dynkin diagram of $x^3 + y^{\ell+1}$

σ_{ij} are the so-called band generators of the braid group.

fundamental groups

Theorem

For $\tilde{X} = \{u \in V_f \mid P(u,0) \neq 0\}$, the discriminant complement:

$$\pi_1 \cong \left\langle t_i,\, i \in I \;\middle|\; \begin{array}{cl} t_i t_j = t_j t_i, & \text{for } {}_i\cdot \quad \cdot_j \\ t_i t_j t_i = t_j t_i t_j, & \text{for } {}_i\cdot\!-\!\!-\!\cdot_j, \\ t_i^{\varepsilon} t_j t_i^{-\varepsilon} t_k = t_k t_i^{\varepsilon} t_j t_i^{-\varepsilon}, & \text{for } {}_i\cdot \cdots \cdot_k,\ {}_j\cdot \end{array} \right\rangle .$$

(ε_{ijk} antisymmetric)

Combinatorial structure

The Dynkin diagram is naturally associated to the geometry of the generic smooth fibre of f:
Vanishing cycles provide a basis for the middle homology and are in bijection to the vertices, edges (in our case) are in bijection to non-zero intersection
(in fact -1).

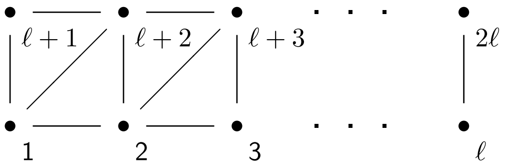


Figure: Dynkin diagram of $x^3 + y^{\ell+1}$

A given geometric basis is naturally acted on by the braid group $\mathrm{Br}_{2\ell}$. Elements in the braid monodromy group act trivially on the Dynkin diagram, since by the theorem of van Kampen they act trivially on the discriminant knot group.

open questions

- Is the braid monodromy group the whole stabiliser group of the Dynkin diagram?
- What results should be expected in the case of arbitrary singularities?
  Gabrielov has Dynkin diagrams with simple edges and triangles, but neither edges of higher multiplicity nor larger cycles.
- How are the invariants related for adjacent singularities?

final remark

with S.Tanabe investigate implications for braid monodromy and fundamental group

- for quotient of base by cyclic group
- for quotient of fibre by cyclic group