Examples

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Gauß

The integral: $\displaystyle\int_\Gamma x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx.$

$$
\begin{aligned}
{}_2F_1\left(\begin{smallmatrix}\alpha,\beta\\ \gamma\end{smallmatrix};z\right) &:= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx \\
&= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n.
\end{aligned}
$$

$$
(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.
$$

The twisted period relation is

$$\begin{aligned}&(1-\gamma+\alpha)(1-\gamma+\beta){}_2F_1\left({\alpha,\beta \atop \gamma};z\right){}_2F_1\left({-\alpha,-\beta \atop 2-\gamma};z\right)\\&-\alpha\beta{}_2F_1\left({\gamma-\alpha-1,\gamma-\beta-1 \atop \gamma};z\right){}_2F_1\left({1-\gamma+\alpha,1-\gamma+\beta \atop 2-\gamma};z\right)\\=&(1-\gamma+\alpha+\beta)(1-\gamma).\end{aligned}$$

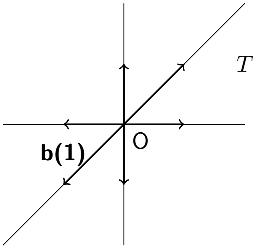
The coefficient of z^n $(n \geq 1)$ is 0.

$$\begin{aligned}&(1-\gamma+\alpha)(1-\gamma+\beta)\sum_{l+m=n}\frac{(\alpha)_l(\beta)_l}{(\gamma)_l(1)_l}\frac{(-\alpha)_m(-\beta)_m}{(2-\gamma)_m(1)_m}\\=&\alpha\beta\sum_{l+m=n}\frac{(\gamma-\alpha-1)_l(\gamma-\beta-1)_l}{(\gamma)_l(1)_l}\frac{(1-\gamma+\alpha)_m(1-\gamma+\beta)_m}{(2-\gamma)_m(1)_m}.\end{aligned}$$

Appell's F_1

We consider a one dimensional integral $f_\Gamma(z) = \int_\Gamma (z_1 + z_4 x)^{-c_1}(z_2 + z_5 x)^{-c_2}(z_3 + z_6 x)^{-c_3} x^{c_4}\omega$. In this case, the A matrix is given by $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ and the parameter vector is $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$. The B matrix is $\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$.

The secondary fan is the following.



We have $\langle \frac{dx}{x}, \frac{dx}{x} \rangle_{ch} = 2\pi\sqrt{-1}\frac{c_1+c_2+c_3}{c_4(c_1+c_2+c_3-c_4)}$. We take $T = \{1234, 2346, 2456\}$ and consider the toric infinity corresponding to it. Taking a restriction to $z_2 = z_3 = z_4 = z_6 = 1$, we obtain an identity for Appell's F_1-series.

$$
\begin{aligned}
&\frac{c_1}{c_4(c_1-c_4)} F_1\left({c_4,c_2,c_3 \atop 1+c_4-c_1}; z_1z_5, z_1\right) F_1\left({-c_4,-c_2,-c_3 \atop 1-c_4+c_1}; z_1z_5, z_1\right) \\
&+\frac{c_3}{(c_1+c_3-c_4)(c_4-c_1)} G_2(c_1,c_2,c_4-c_1,c_1+c_3-c_4;-z_1,-z_5)\times \\
&G_2(-c_1,-c_2,c_1-c_4,c_4-c_1-c_3;-z_1,-z_5) \\
&+\tfrac{c_2}{(c_1+c_2+c_3-c_4)(c_4-c_1-c_3)} F_1\left({c_1+c_2+c_3-c_4,c_1,c_3 \atop 1+c_1+c_3-c_4}; z_1z_5, z_5\right)\times \\
&F_1\left({c_4-c_1-c_2-c_3,-c_1,-c_3 \atop 1+c_4-c_1-c_3}; z_1z_5, z_5\right) \\
=&\frac{c_1+c_2+c_3}{c_4(c_1+c_2+c_3-c_4)}
\end{aligned}
$$

Here, we have put

$$F_1\left({a,b,b' \atop c}; x, y\right) = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n$$

$$G_2(a,a',b,b';x,y) = \sum_{m,n\geq 0} \frac{(a)_m(a')_n(b)_{n-m}(b')_{m-n}}{m!n!} x^m y^n.$$

木村-原岡-高野 (Kimura-Haraoka-Takano)

$$f_\Gamma(z) = \int_\Gamma \prod_{j=3}^{4} (z_{0j} + z_{1j}x_1 + z_{2j}x_2)^{-c_j} e^{z_{15}x_1 + z_{25}x_2} x_1^{c_1 - 1} x_2^{c_2 - 1} dx_1 dx_2$$

$$z = \begin{pmatrix} z_{03} & z_{04} & * \\ z_{13} & z_{14} & z_{15} \\ z_{23} & z_{24} & z_{25} \end{pmatrix}$$

$$c_0 = c_3 + c_4 - c_1 - c_2$$

The twisted period relation is

$$c_1 c_2 c_3 c_4 \sum_{i=1}^{6} \frac{\pi^4}{\sin \pi(v_i)} \varphi_i(z;c) \varphi_i^{\vee}(z;c) = 1.$$

$$\begin{aligned}
v_1 &= {}^t(-c_3, -c_4, c_0 + c_1, -c_1) \\
v_2 &= {}^t(-c_3, -c_2 + c_3, -c_0 - c_1, c_0) \\
v_3 &= {}^t(-c_3, -c_2 + c_3, -c_1, -c_0) \\
v_4 &= {}^t(-c_2, c_2 - c_3, -c_4, c_0) \\
v_5 &= {}^t(-c_2, c_2 - c_3, c_0 - c_4, -c_0) \\
v_6 &= {}^t(-c_2, -c_1, -c_0 + c_4, -c_4)
\end{aligned}$$

$$
\begin{aligned}
&\varphi_1(z;c)\\
=&z_{23}^{-c_3} z_{24}^{-c_4} z_{25}^{c_0+c_1} z_{15}^{-c_1}\\
&\sum_{u_{13},u_{14},u_{03},u_{04}\geq 0} \frac{1}{\Gamma(1-c_3-u_{13}-u_{03})\Gamma(1-c_4-u_{14}-u_{04})}\\
&\frac{1}{\Gamma(1+c_0+c_1+u_{13}+u_{14}+u_{03}+u_{04})\Gamma(1-c_1-u_{13}-u_{14})}\\
&\frac{(z_{23}^{-1} z_{25} z_{15}^{-1} z_{13})^{u_{13}} (z_{24}^{-1} z_{25} z_{15}^{-1} z_{14})^{u_{14}} (z_{23}^{-1} z_{25} z_{03})^{u_{03}} (z_{24}^{-1} z_{25} z_{04})^{u_{04}}}{u_{13}! u_{14}! u_{03}! u_{04}!}
\end{aligned}
$$

$\varphi_1^\vee(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

$$
\begin{aligned}
&\varphi_2(z;c) \\
=&z_{23}^{-c_3} z_{24}^{-c_2+c_3} z_{14}^{-c_0-c_1} z_{15}^{c_0} \\
&\sum_{u_{25},u_{13},u_{03},u_{04}\geq 0} \frac{1}{\Gamma(1-c_3-u_{13}-u_{03})\Gamma(1-c_2+c_3-u_{25}+u_{13}+u_{03})} \\
&\frac{1}{\Gamma(1-c_0-c_1+u_{25}-u_{13}-u_{03}-u_{04})\Gamma(1+c_0-u_{25}+u_{03}+u_{04})} \\
&\frac{(z_{24}^{-1} z_{14} z_{15}^{-1} z_{25})^{u_{25}} (z_{23}^{-1} z_{24} z_{14}^{-1} z_{13})^{u_{13}} (z_{23}^{-1} z_{24} z_{14}^{-1} z_{15} z_{03})^{u_{03}} (z_{14}^{-1} z_{15} z_{04})^{u_{0}}}{u_{25}!u_{13}!u_{03}!u_{04}!}
\end{aligned}
$$

$\varphi_2^{\vee}(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

$$
\begin{aligned}
&\varphi_3(z;c)\\
=&z_{23}^{-c_3} z_{24}^{-c_2+c_3} z_{14}^{-c_1} z_{04}^{-c_0}\\
&\sum_{u_{25},u_{15},u_{13},u_{03}\geq 0} \frac{1}{\Gamma(1-c_3-u_{13}-u_{03})\Gamma(1-c_2+c_3-u_{25}+u_{13}+u_{03})}\\
&\frac{1}{\Gamma(1-c_1-u_{15}-u_{13})\Gamma(1-c_0+u_{25}+u_{15}-u_{03})}\\
&\frac{(z_{24}^{-1}z_{04}z_{25})^{u_{25}}(z_{14}^{-1}z_{04}z_{15})^{u_{15}}(z_{23}^{-1}z_{24}z_{14}^{-1}z_{13})^{u_{13}}(z_{23}^{-1}z_{24}z_{04}^{-1}z_{03})^{u_{03}}}{u_{25}!u_{15}!u_{13}!u_{03}!}
\end{aligned}
$$

$\varphi_3^{\vee}(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

$$\begin{aligned}
&\varphi_4(z;c)\\
=&z_{23}^{-c_2} z_{13}^{c_2-c_3} z_{14}^{-c_4} z_{15}^{c_0}\\
&\sum_{u_{24},u_{25},u_{03},u_{04}\geq 0} \frac{1}{\Gamma(1-c_2-u_{24}-u_{25})\Gamma(1+c_2-c_3+u_{24}+u_{25}-u_{03})}\\
&\frac{1}{\Gamma(1-c_4-u_{24}-u_{04})\Gamma(1+c_0-u_{25}+u_{03}+u_{04})}\\
&\frac{(z_{23}^{-1}z_{13}z_{14}^{-1}z_{24})^{u_{24}}(z_{23}^{-1}z_{13}z_{15}^{-1}z_{25})^{u_{25}}(z_{13}^{-1}z_{15}z_{03})^{u_{03}}(z_{14}^{-1}z_{15}z_{04})^{u_{04}}}{u_{24}!u_{25}!u_{03}!u_{04}!}
\end{aligned}$$

$\varphi_4^\vee(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

$$
\begin{aligned}
&\varphi_5(z;c)\\
=&z_{23}^{-c_2} z_{13}^{c_2-c_3} z_{14}^{c_0-c_4} z_{04}^{-c_0}\\
&\sum_{u_{24},u_{23},u_{15},u_{03}\geq 0} \frac{1}{\Gamma(1-c_2-u_{24}-u_{25})\Gamma(1+c_2-c_3+u_{24}+u_{25}-u_{03})}\\
&\frac{1}{\Gamma(1+c_0-c_4-u_{24}-u_{25}-u_{15}+u_{03})\Gamma(1-c_0+u_{25}+u_{15}-u_{03})}\\
&\frac{(z_{23}^{-1}z_{13}z_{14}^{-1}z_{24})^{u_{24}}(z_{23}^{-1}z_{13}z_{14}^{-1}z_{04}z_{25})^{u_{25}}(z_{14}^{-1}z_{04}z_{15})^{u_{15}}(z_{13}^{-1}z_{14}z_{04}^{-1}z_{03})^{u_{0}}}{u_{24}!u_{25}!u_{15}!u_{03}!}
\end{aligned}
$$

$\varphi_5^{\vee}(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

$$
\begin{aligned}
&\varphi_6(z;c) \\
&= z_{23}^{-c_2} z_{13}^{-c_1} z_{03}^{-c_0+c_4} z_{04}^{-c_4} \\
&\quad \sum_{u_{24},u_{25},u_{14},u_{15}\geq 0} \frac{1}{\Gamma(1-c_2-u_{24}-u_{25})\Gamma(1-c_1-u_{14}-u_{15})} \\
&\frac{1}{\Gamma(1-c_0+c_4+u_{24}+u_{25}+u_{14}+u_{15})\Gamma(1-c_4-u_{24}-u_{14})} \\
&\frac{(z_{23}^{-1}z_{03}z_{04}^{-1}z_{24})^{u_{24}}(z_{23}^{-1}z_{03}z_{25})^{u_{25}}(z_{13}^{-1}z_{03}z_{04}^{-1}z_{14})^{u_{14}}(z_{13}^{-1}z_{03}z_{15})^{u_{15}}}{u_{24}!u_{25}!u_{14}!u_{15}!}
\end{aligned}
$$

$\varphi_6^\vee(z;c)$ is given by $c_i \to -c_i$ and $z_{15}, z_{25} \to -z_{15}, -z_{25}$ in the formula above.

Note that if we substitute

$$\begin{pmatrix} z_{03} & z_{04} & * \\ z_{13} & z_{14} & z_{15} \\ z_{23} & z_{24} & z_{25} \end{pmatrix} = \begin{pmatrix} 1 & 1 & * \\ 1 & \zeta_1 & \zeta_1\zeta_2 \\ 1 & \zeta_1\zeta_3 & \zeta_1\zeta_2\zeta_3\zeta_4 \end{pmatrix},$$

all the Laurent series $\varphi_i(z;c)$ and $\varphi_i^\vee(z;c)$ above become power series, i. e., they do not contain any negative power.

Hypergeometric series at the torus fixed point

$\sigma \subset \{1, \dots, N\}$: an $(n+k)$-dimensional simplex, i.e., the square matrix $A_\sigma = (\mathbf{a}(j))_{j \in \sigma}$ is invertible.

$$\delta := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \\ c \end{pmatrix}$$

$$\varphi_\sigma(z) := z_\sigma^{-A_\sigma^{-1}\delta} \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{\bar{\sigma}}} \frac{(z_\sigma^{-A_\sigma^{-1} A_{\bar{\sigma}}} z_{\bar{\sigma}})^{\mathbf{m}}}{\Gamma(\mathbf{1}_\sigma - A_\sigma^{-1}(\delta + A_{\bar{\sigma}}\mathbf{m}))\mathbf{m}!}$$

Theorem (M.-H. ArXiv1904.00565)

Suppose that four vectors $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^{n\times 1}, \mathbf{b}, \mathbf{b}' \in \mathbb{Z}^{k\times 1}$ *and a unimodular regular triangulation T are given. If the parameter* d *is generic, one has an identity*

$$\frac{\langle x^{\mathbf{a}} h^{\mathbf{b}} \frac{dx}{x}, x^{\mathbf{a}'} h^{\mathbf{b}'} \frac{dx}{x} \rangle_{ch}}{(2\pi\sqrt{-1})^n}$$
$$=(-1)^{|\mathbf{b}|+|\mathbf{b}'|}\gamma_1 \cdots \gamma_k (\gamma - \mathbf{b})_{\mathbf{b}} (-\gamma - \mathbf{b}')_{\mathbf{b}'} \times$$
$$\sum_{\sigma \in T} \frac{\pi^{n+k}}{\sin \pi A_\sigma^{-1}\delta} \varphi_\sigma \left(z; \begin{pmatrix} \gamma - \mathbf{b} \\ c + \mathbf{a} \end{pmatrix} \right) \varphi_\sigma \left(z; \begin{pmatrix} -\gamma - \mathbf{b}' \\ -c + \mathbf{a}' \end{pmatrix} \right)$$

near the torus fixed point corresponding to T*. Here,* $\frac{dx}{x} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$, $(\gamma - \mathbf{b})_{\mathbf{b}} = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \mathbf{b})}$*, and* $(-\gamma - \mathbf{b}')_{\mathbf{b}'} = \frac{\Gamma(-\gamma)}{\Gamma(-\gamma - \mathbf{b}')}$