Monodromy representations for several hypergeometric systems by virtue of intersection forms

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1. Introduction

There are several hypergeometric systems whose solutions admit Euler type integrals. It is classically known that we can obtain a circuit transformation along a loop for each of such systems by deforming areas of integration and following changes of branches of an integrand on them along this loop.

However, it is difficult to trace these for higher dimensional cases. In this talk, I give a way to resolve this difficulty.

I utilize twisted homology groups and express circuit transformations as reflections with respect to the intersection form between them.

I illustrate this method by using Lauricella's hypergeometric system F_D of rank $m+1$ in m-variables.

I introduce some results for the monodromy representation of Lauricella's F_A (m variables, rank 2^m) studied in

[MY] Matsumoto K. and Yoshida M.,
Monodromy of Lauricella's hypergeometric F_A-system, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **13** (2014), 551–577,

and for that of Lauricella's F_C (m-variables, rank 2^m) studied in

[G] Goto Y.,
The monodromy representation of Lauricella's hypergeometric function F_C, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **16** (2016), 1409–1445,

and for the Aomoto-Gelfand hypergeometric system (km-variables, rank $(k+m)!/(k!m!)$) in a recent joint work with Terasoma T.

2. Lauricella's hypergeometric system $\mathcal{F}_D$

The Gauss hypergeometric series is defined by

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} x^n, \qquad \begin{array}{l} \{x \in \mathbb{C} \mid |x| < 1\} \\ c \neq 0, -1, -2, \dots, \\ (a,n) = \Gamma(a+n)/\Gamma(a). \end{array}$$

It admits an Euler type integral

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^{\infty} t^{b-c}(t-1)^{c-a}(t-x)^{-b} \frac{dt}{t-1},$$

$(\mathrm{Re}(c) > \mathrm{Re}(a) > 0)$ and satisfies hypergeometric differential equation

$$\left[x(1-x)(\frac{d}{dx})^2 + \{c-(a+b+1)x\}(\frac{d}{dx}) - ab\right] f = 0. \qquad \text{(HGDE)}$$

Lauricella's hypergeometric system $\mathcal{F}_D$ is known to be the simplest one of multi-variables versions of (HGDE).

Lauricella's hypergeometric series is define by

$$F_D(a,b,c;x) = \sum_{n\in\mathbb{N}^m} \frac{(a,\sum_{i=1}^m n_i)\prod_{i=1}^m (b_i,n_i)}{(c,\sum_{i=1}^m n_i)\prod_{i=1}^m (1,n_i)} \prod_{i=1}^m x_i^{n_i},$$

where $x=(x_1,\dots,x_m)$ are main variables, and a, $b=(b_1,\dots,b_m)$, c are complex parameters with $c\notin -\mathbb{N}=\{0,-1,-2,\dots\}$, and it converges absolutely and uniformly on any compact set in

$$\mathbb{D}=\left\{x\in\mathbb{C}^m \,\middle|\, \max_{1\le i\le m}|x_i|<1\right\}.$$

It admits an Euler type integral

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_1^\infty t^{\Sigma_i b_i - c}(t-1)^{c-a}\prod_{i=1}^m (t-x_i)^{-b_i}\frac{dt}{t-1} \tag{EI}$$

$(\mathrm{Re}(c)>\mathrm{Re}(a)>0)$.

$F_D(a, b, c; x)$ is annihilated by differential operators

$$x_i(1-x_i)\partial_i^2 + (1-x_i)\sum_{1\le j\le m}^{j\ne i} x_j\partial_i\partial_j \qquad (1 \le i \le m)$$
$$+[c-(a+b_i+1)x_i]\partial_i - b_i\sum_{1\le j\le m}^{j\ne i} x_j\partial_j - ab_i,$$

$$(x_i - x_j)\partial_i\partial_j - b_j\partial_i + b_i\partial_j, \qquad (1 \le i < j \le m)$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Lauricella's hypergeometric system $\mathcal{F}_D$ is the system generated by these differential equations.

Set $x_0 = 0$, $x_{m+1} = 1$, and define

$$S_{ij} = \{x \in \mathbb{C}^m \mid x_i = x_j\} \quad (0 \le i < j \le m+1) \quad (S_{0,m+1} = \phi).$$

$$S = \bigcup_{0\le i<j\le m} S_{ij}, \quad X = \mathbb{C}^m - S.$$

Fact 2.1

$\mathcal{F}_D$ is of rank $m+1$, and its singular locus is S.
That means that for ${}^\forall x \in X$, ${}^\exists U_x$: a nbd. of x s.t. the space $Sol_{\mathcal{F}_D}(U_x)$ of single-valued holomorphic solutions to $\mathcal{F}_D$ on U_x is $(m+1)$-dim. vector space, and if $x \in S$ then $\dim Sol_{\mathcal{F}_D}(U_x) \leq m$ for ${}^\forall U_x$.

$\dot{U}$: a nbd. of $\dot{x} = (\frac{1}{m+1}, \dots, \frac{m}{m+1}) \in X$ s.t. $\dim Sol_{\mathcal{F}_D}(\dot{U}) = m+1$.
A loop ρ with terminal $\dot{x}$ causes a linear transformation of $Sol_{\mathcal{F}_D}(\dot{U})$ by the analytic continuation along ρ.

It induces a group homomorphism

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL(Sol_{\mathcal{F}_D}(\dot{U})),$$

which is called monodromy representation. Its image is called the monodromy group, and $\mathcal{M}(\rho)$ is called the circuit transformation along ρ.

ρ_{ij} $(1 \leq i < j \leq m+1)$: a loop in X starting from $\dot{x}$, fixing $x_k = \dot{x}_k$ for $k \neq i$, approaching $\dot{x}_j$ via the upper half space of the x_i-space, turning around this point positively, and tracing back. Set ρ_{0j} $(1 \leq j \leq m)$ by exchanging the roles of $i = 0$ and j.

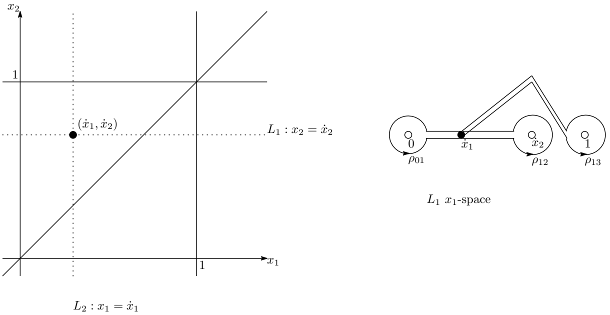


Figure: Loops

Fact 2.2

Loops ρ_{ij} $(0 \leq i < j \leq m+1,\ (i,j) \neq (0, m+1))$ generate $\pi_1(X, \dot{x})$.

Problem 2.3

Express the circuit transform $\mathcal{M}_{ij} = \mathcal{M}(\rho_{ij})$.

Introduce parameters α_i $(i = 0, 1, \ldots, m, m+1, m+2)$ by

$$\alpha_0 = (\sum_{i=1}^{m} b_i) - c,\ \ \alpha_i = -b_i\ (i = 1, \ldots, m),\ \ \alpha_{m+1} = c - a,\ \ \alpha_{m+2} = a,$$

and assume non-integral conditions (NIC) on them:

$$\alpha_0,\ \alpha_1, \ldots, \alpha_m,\ \alpha_{m+1},\ \alpha_{m+2} \notin \mathbb{Z}. \tag{NIC}$$

In case of $m = 1$, this assumption is $a, b, c-a, c-b \notin \mathbb{Z}$ (permitting $c \in \mathbb{Z}$).

3. Twisted homology groups

We explain twisted homology groups and the intersection form among them.

Note that solutions to $\mathcal{F}_D$ can be expressed by

$$\int_\gamma u(t,x)\varphi,$$

$$u = u(t,x) = t^{\alpha_0}(t-x_1)^{\alpha_1}\cdots(t-x_m)^{\alpha_m}(t-1)^{\alpha_{m+1}}, \quad \varphi = \frac{dt}{t-1}.$$

Fix x and set $\mathbb{C}_x = \mathbb{C} - \{0, x_1, \dots, x_m, 1\}$. Though u is multi-valued on $\mathbb{C}_x$, a branch of u on a 1-simplex γ is uniquely determined. We consider the pair σ^u of a k-simplex σ in $\mathbb{C}_x$ and a branch of u on σ. Let $\mathcal{C}_k(u)$ be the vector space of formal linear combinations of σ^u's over $\mathbb{C}$.

A twisted boundary operator is

$$\partial^u : \mathcal{C}_k(u) \ni \sigma^u \mapsto (\partial\sigma)^{u|_{\partial\sigma}} \in \mathcal{C}_{k-1}(u),$$

where ∂ is the usual boundary, $u|_{\partial\sigma}$ is the restriction of u to $\partial\sigma$. Define the twisted homology group $H_1(\mathbb{C}_x, u)$ by

$$H_1(\mathbb{C}_x, u) = \ker(\partial_u : \mathcal{C}_1(u) \to \mathcal{C}_0(u))/\partial_u(\mathcal{C}_2(u)).$$

Fact 3.1

Under (NIC), $\dim H_1(\mathbb{C}_x, u) = m + 1$, and the linear map

$$H^0(\prod_{x\in U_x} H_1(\mathbb{C}_x, u)) \ni \sigma^u \mapsto \int_\sigma u(t,x)\varphi \in Sol_{\mathcal{F}_D}(U_x),$$

is isomorphic, where $\prod_{x\in U_x} H_1(\mathbb{C}_x, u)$ is the trivial bundle of fibers $H_1(\mathbb{C}_x, u)$ over a nbd. U_x of x, $H^0(\prod_{x\in U_x} H_1(\mathbb{C}_x, u))$ is the space of its sections, which are determined by elements $\gamma^u \in H_1(\mathbb{C}_x, u)$ over x.

For $u^{-1} = 1/u$, a twisted homology group $H_1(\mathbb{C}_x, u^{-1})$ is defined . There is the intersection form $\mathcal{I}$ between $H_1(\mathbb{C}_x, u)$, $H_1(\mathbb{C}_x, u^{-1})$.
Suppose that 1-simplexes γ_+, γ_- intersect transversally at finitely many points p_i's. Then we set

$$(\gamma_+^u \cdot \gamma_-^{u^{-1}}) = \sum_i (\gamma_+ \cdot \gamma_-)_{p_i} \cdot u_{\gamma_+}(p_i) \cdot u_{\gamma_-}^{-1}(p_i),$$

where $(\gamma_+ \cdot \gamma_-)_{p_i}$ is the top. intersection number of γ_+ and γ_- at p_i.
The intersection form $\mathcal{I}$ is define by its linear extension.

Fact 3.2 ([AoK, §2.3.3], [Y, Chap. IV, §7])

Let γ_{ij} be a path in $\mathbb{C}_{\dot{x}}$ from $\dot{x}_i$ to $\dot{x}_j$ via the upper half space of $\mathbb{C}_x$, and set $\lambda_i = \exp(2\pi\sqrt{-1}\alpha_i)$. Then

$$\mathcal{I}(\gamma_{ij}^u, \gamma_{pq}^{u^{-1}}) = \begin{cases} \frac{1-\lambda_i\lambda_j}{(1-\lambda_i)(1-\lambda_j)} & \text{if} \quad (i,j)=(p,q), \\ \frac{\lambda_i}{1-\lambda_i} & \text{if} \quad i=p,\ j>q, \\ \frac{-\lambda_j}{1-\lambda_j} & \text{if} \quad j=p, \\ \frac{1}{1-\lambda_j} & \text{if} \quad i<p,\ j=q, \\ 1 & \text{if} \quad i<p<j<q, \\ 0 & \text{if} \quad i<p<q<j,\ i<j<p<q. \end{cases}$$

For other cases, use formula $\mathcal{I}(\gamma_{pq}^u, \gamma_{ij}^{u^{-1}}) = -\mathcal{I}(\gamma_{ij}^u, \gamma_{pq}^{u^{-1}})^\vee$, where $^\vee$ is an operator changing λ_k into $1/\lambda_k$ $(0 \le k \le m+2)$.

Set $\gamma_0 = \gamma_{m+1,m+2}$, $\gamma_1 = \gamma_{m+1,1}$, ..., $\gamma_m = \gamma_{m+1,m}$ (γ_0 is a path from $t=1$ to $t=\infty$, γ_i is a path from $t=1$ to $t=\dot{x}_i$ via the upper half space of $\mathbb{C}_{\dot{x}}$).

Corollay 3.3

Under (NIC), $\gamma_0^u, \gamma_1^u, \ldots, \gamma_m^u$ is a basis of $H_1(\mathbb{C}_{\dot{x}}, u)$.

Corollay 3.4

The intersection matrix $H = \left[\mathcal{I}(\gamma_i^u, \gamma_j^{u^{-1}})\right]_{0 \le i,j \le m}$ is

$$
\begin{aligned}
H \quad = \quad & \frac{1}{1-\lambda_{m+1}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{m+1} & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_{m+1} & \cdots & \lambda_{m+1} & 1 \end{pmatrix} \\
& + \mathrm{diag}\left(\frac{\lambda_{m+2}}{1-\lambda_{m+2}}, \frac{\lambda_1}{1-\lambda_1}, \ldots, \frac{\lambda_m}{1-\lambda_m}\right),
\end{aligned}
$$

and

$$
\det(H) = \frac{1-\lambda_0^{-1}}{\prod_{i=1}^{m+2}(1-\lambda_i)} \neq 0.
$$

4. Monodromy representation of $\mathcal{F}_D$

By patching trivial vector bundles $\displaystyle\prod_{x\in U_x} H_1(\mathbb{C}_x, u)$, we have a local system

$$\mathcal{H}_1(\mathbb{C}_x, u) = \bigcup_{U_x \subset X} \Big[\prod_{x\in U_x} H_1(\mathbb{C}_x, u) \Big]$$

over X of fiber $H_1(\mathbb{C}_x, u)$.

By Fact 3.1, the monodromy representation $\mathcal{M}$ of $\mathcal{H}_1(\mathbb{C}_x, u)$ is equivalent to that of $\mathcal{F}_D$. We use the same notations for the monodromy representation of $\mathcal{H}_1(\mathbb{C}_x, u)$ as those of $\mathcal{F}_D$.

Lemma 4.1

The twisted cycle γ^u_{ij} is a $\lambda_i\lambda_j$-eigenvector of $\mathcal{M}_{ij}$, i.e.,

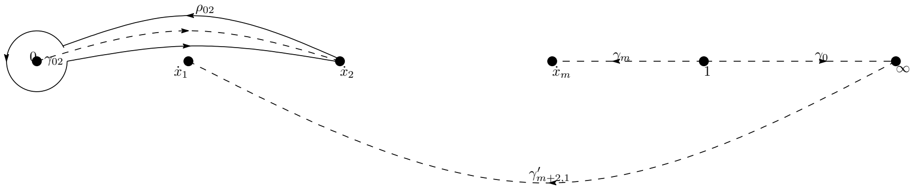
$$\mathcal{M}_{ij}(\gamma^u_{ij}) = (\lambda_i\lambda_j)\gamma^u_{ij}.$$

The 1-eigenspace of $\mathcal{M}_{ij}$ is m-dim.

Remark 4.2

If $\alpha_i + \alpha_j \in \mathbb{Z}$ then the $\lambda_i\lambda_j$-eigenvector γ_{ij}^u of $\mathcal{M}_{ij}$ belongs to the 1-eigenspace of $\mathcal{M}_{ij}$. In this case, $\mathcal{M}_{ij}$ is not diagonalizable.

Proof. The loop ρ_{ij} causes deformations of paths γ's defining twisted cycles γ^u. Note that γ_{ij} is invariant under the deformation caused by ρ_{ij}, see Figure below.



$\gamma_{m+2,1} \neq \gamma'_{m+2,1}$, $\mathbb{I}_h({\gamma'_{m+2,1}}^u, \gamma_{02}^{u^{-1}}) = 0$

To follow the change of the branch of $u(t,x)$ on γ_{ij}, express the path γ_{ij} as a map from the interval $(0,1)$ to $\mathbb{C}_x$, and consider the pull back $\gamma_{ij}^*(u(t,x))$ to $(0,1)$. Compute the change of $\gamma_{ij}^*(u(t,x))$ by expressing the loop ρ_{ij} as a map from $[0,1]$ to X. Thus

$$\mathcal{M}_{ij}(\gamma_{ij}^u) = (\lambda_i\lambda_j)\gamma_{ij}^u.$$

Consider m paths $\gamma'_{m+2,k}$ from $\dot{x}_{m+2} = \infty$ to $\dot{x}_k$ via the lower half space for $0 \leq k \leq m+1$ $k \neq i,j$.

Note that $\gamma'_{m+2,k}$ are not involved in the deformation caused by ρ_{ij}. The branch of $u(t,x)$ on $\gamma'_{m+2,k}$ is invariant under this deformation. Thus

$$\mathcal{M}_{ij}({\gamma'_{m+2,k}}^u) = {\gamma'_{m+2,k}}^u.$$

Since $\det\left(\mathcal{I}({\gamma'_{m+2,k}}^u, {\gamma'_{m+2,l}}^{u^{-1}})\right)_{k,l} \neq 0$ by Fact 3.2, these m twisted cycles are linearly independent. $\square$

Remark 4.3

To get $\mathcal{M}_{ij}$, we have only to consider the deformation γ^u_{ij}, which vanishes along the approach of x_i to x_j for ρ_{ij}. This proof is not based on figures, and valid in higher dimensional cases.

Theorem 4.4 (Monodromy priciple for the intersection form)

(1) $\mathcal{I}(\mathcal{M}_\rho(\gamma^u_+), \mathcal{M}^\vee_\rho(\gamma^{u^{-1}}_-)) = \mathcal{I}(\gamma^u_+, \gamma^{u^{-1}}_-), \ \ {}^\forall\gamma^{u^{\pm 1}}_\pm \in H_1(\mathbb{C}_x, u^\pm)$.

(2) *Let* M_ρ *be the representation matrix of* $\mathcal{M}_\rho$ *w.r.t. the basis* $(\gamma^u_0, \gamma^u_1, \ldots, \gamma^u_m)$, *i.e.,* $\mathcal{M}_\rho(\gamma^u_0, \gamma^u_1, \ldots, \gamma^u_m) = (\gamma^u_0, \gamma^u_1, \ldots, \gamma^u_m)M_\rho$. *Then* ${}^tM_\rho\, H\, M^\vee_\rho = H, \ \ {}^tM^\vee_\rho\ {}^tH\, M_\rho = {}^tH$.

(3) *Let* γ^u *be a* β*-eigenvector of* $\mathcal{M}_\rho$*. Then*

$$\mathcal{I}(\gamma^u, (\gamma^u)^\vee) \neq 0 \Rightarrow \beta \cdot \beta^\vee = 1.$$

(4) *Let* γ^u_1 *and* γ^u_2 *be a* β_1*-eigenvector and a* β_2*-eigenvector of* $\mathcal{M}_\rho$*, respectively. Then*

$$\beta_1 \cdot \beta^\vee_2 \neq 1 \Rightarrow \mathcal{I}(\gamma^u_1, (\gamma^u_2)^\vee) = 0.$$

Proof. (1) γ_+^u and $\gamma_-^{u^{-1}}$ are naturally continued to any fiber over x in $U_{\dot{x}}$, and their intersection number is stable. Since any intersection number is locally constant, it is stable under the continuation along ρ.

(2) Express γ^u by a linear combination of $(\gamma_0^u, \dots \gamma_m^u)$ as

$(\gamma_0^u, \dots, \gamma_m^u) \begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix}$ $(g_0, \dots, g_m \in \mathbb{C})$, then $\mathcal{M}_\rho(\gamma^u)$ and $\mathcal{I}_h(\gamma^u, (\gamma^u)^\vee)$ are

$$(\gamma_0^u, \dots, \gamma_m^u) M_\rho \begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix}, \quad (g_0, \dots, g_m) H \begin{pmatrix} g_0^\vee \\ \vdots \\ g_m^\vee \end{pmatrix}.$$

Thus

$$\mathcal{I}_h(\mathcal{M}_\rho(\gamma^u), (\mathcal{M}_\rho(\gamma^u))^\vee) = (g_0, \dots, g_m)\, {}^t M_\rho H M_\rho^\vee \begin{pmatrix} g_0^\vee \\ \vdots \\ g_m^\vee \end{pmatrix}.$$

(1) yields that

$$({g}_0,\ldots,{g}_m)\ {}^t M_\rho H M_\rho^\vee \begin{pmatrix} g_0^\vee \\ \vdots \\ g_m^\vee \end{pmatrix} = ({g}_0,\ldots,{g}_m) H \begin{pmatrix} g_0^\vee \\ \vdots \\ g_m^\vee \end{pmatrix},$$

for any $g_0, \cdots g_m$. Thus ${}^t M_\rho\ H\ M_\rho^\vee = H$.

(3) Since $\mathcal{M}_\rho(\gamma^u) = \beta\gamma^u$, (1) yields that

$$\begin{aligned} \mathcal{I}(\gamma^u, (\gamma^u)^\vee) &= \mathcal{I}(\mathcal{M}_\rho(\gamma^u), \mathcal{M}_\rho^\vee((\gamma^u)^\vee)) = \mathcal{I}(\beta\gamma^u, \beta^\vee(\gamma^u)^\vee) \\ &= (\beta\cdot\beta^\vee)\cdot\mathcal{I}(\gamma^u, (\gamma^u)^\vee). \end{aligned}$$

Thus $\mathcal{I}(\gamma^u, (\gamma^u)^\vee) \neq 0 \Rightarrow \beta\cdot\beta^\vee = 1$.

(4) (1) yields that

$$\begin{aligned} \mathcal{I}(\gamma_1^u, (\gamma_2^u)^\vee) &= \mathcal{I}(\mathcal{M}_\rho(\gamma_1^u), \mathcal{M}_\rho^\vee((\gamma_2^u)^\vee)) = \mathcal{I}(\beta_1\gamma_1^u, \beta_2^\vee(\gamma_2^u)^\vee) \\ &= (\beta_1\cdot\beta_2^\vee)\cdot\mathcal{I}(\gamma_1^u, (\gamma_2^u)^\vee). \end{aligned}$$

Thus $\beta_1\cdot\beta_2^\vee \neq 1 \Rightarrow \mathcal{I}(\gamma_1^u, (\gamma_2^u)^\vee) = 0$.

Remark 4.5

If $\mathcal{I}(\gamma_2^u, (\gamma_2^u)^\vee) \neq 0$ then $\beta_2^\vee = 1/\beta_2$; in this case, the condition $\beta_1 \cdot \beta_2^\vee \neq 1$ in Theorem 4.4 (4) is equivalent to $\beta_1 \neq \beta_2$.

Theorem 4.6

The circuit transformation $\mathcal{M}_{ij}$ is expressed by

$$\begin{aligned}\gamma^u \mapsto \gamma^u &- \gamma_{ij}^u(1-\lambda_i\lambda_j)\mathcal{I}(\gamma_{ij}^u, (\gamma_{ij}^u)^\vee)^{-1}\mathcal{I}(\gamma^u, (\gamma_{ij}^u)^\vee) \\ &= \gamma^u - \gamma_{ij}^u(1-\lambda_i)(1-\lambda_j)\mathcal{I}(\gamma^u, (\gamma_{ij}^u)^\vee).\end{aligned}$$

Proof. Let $\mathcal{M}'_{ij}$ be the first expression of this theorem. Assume $\lambda_i\lambda_j \neq 1$ temporally. Note that

$$\mathcal{M}'_{ij}(\gamma_{ij}^u) = \gamma_{ij}^u - \gamma_{ij}^u(1-\lambda_i\lambda_j)\mathcal{I}(\gamma_{ij}^u, (\gamma_{ij}^u)^\vee)\mathcal{I}(\gamma_{ij}^u, (\gamma_{ij}^u)^\vee)^{-1} = \gamma_{ij}^u\lambda_i\lambda_j,$$
$$\mathcal{M}'_{ij}(\gamma^u) = \gamma^u - \gamma_{ij}^u(1-\lambda_i\lambda_j)\mathcal{I}(\gamma^u, (\gamma_{ij}^u)^\vee)\mathcal{I}(\gamma_{ij}^u, (\gamma_{ij}^u)^\vee)^{-1} = \gamma^u,$$

for any γ^u satisfying $\mathcal{I}(\gamma^u, (\gamma_{ij}^u)^\vee) = 0$.

Thus γ_{ij}^u is a $\lambda_i\lambda_j$-eigenvector of $\mathcal{M}'_{ij}$, and the 1-eigenspace of $\mathcal{M}'_{ij}$ is $\{\gamma^u \in H_1(\mathbb{C}_{\dot{x}}, u) \mid \mathcal{I}(\gamma^u, (\gamma_{ij}^u)^\vee) = 0\}$. By Lemma 4.1 and Theorem 4.4, this space coincide with the 1-eigenspace of $\mathcal{M}_{ij}$. Coincidence of the eigenspaces of $\mathcal{M}'_{ij}$ and $\mathcal{M}_{ij}$ yields $\mathcal{M}'_{ij} = \mathcal{M}_{ij}$.

Since $\mathcal{I}(\gamma_{ij}^u, (\gamma_{ij}^u)^\vee) = \dfrac{1-\lambda_i\lambda_j}{(1-\lambda_i)(1-\lambda_j)}$ by Fact 3.2, we have the second expression, which is valid even in the case $\lambda_i\lambda_j = 1$. □

Remark 4.7

The expressions $\mathcal{M}_{ij}$ in Theorem 4.6 is independent of a basis of $H_1(\mathbb{C}_{\dot{x}}, u)$. The first one is a complex reflection of root γ_{ij}^u w.r.t. $\mathcal{I}$. Thanks to the intersection form, we can get $\mathcal{M}_{ij}$ only to specify the vanishing cycle γ_{ij}^u.

By using a basis $(\gamma_0^u, \gamma_1^u, \dots, \gamma_m^u)$ of $H_1(\mathbb{C}_{\dot{x}}, u)$, we represent $\mathcal{M}_{ij}$ by a matrix M_{ij}.

Corollay 4.8

M_{ij} is expressed by the intersection matrix H as

$$
\begin{aligned}
M_{ij} &= \mathrm{id}_{m+1} - (1-\lambda_i\lambda_j) w_{ij} ({}^t w_{ij}^\vee\, {}^t H w_{ij})^{-1}\, {}^t w_{ij}^\vee\, {}^t H, \\
&= \mathrm{id}_{m+1} - (1-\lambda_i)(1-\lambda_j) w_{ij}\, {}^t w_{ij}^\vee\, {}^t H.
\end{aligned}
$$

where column vectors w_{ij} are

$$
\begin{aligned}
w_{0j} &= \tilde{e}_j + \frac{(1-\lambda_{m+2})}{\lambda_{m+2}(1-\lambda_0)} \tilde{e}_0 + \sum_{k=1}^{m} \frac{(\lambda_0 \cdots \lambda_{k-1})(1-\lambda_k)}{1-\lambda_0} \tilde{e}_k, \\
w_{ij} &= \tilde{e}_i - \tilde{e}_j \quad (1 \le i < j \le m), \\
w_{i,m+1} &= \tilde{e}_i = {}^t \begin{array}{c} \\ \end{array} \begin{array}{ccccccccc} 0\text{-}th & & & & i\text{-}th & & & & \\ \end{array} \\
& \;\; {}^t\begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}.
\end{aligned}
$$

Proof. Note that

$$\gamma_{ij}^{u} = (\gamma_0^u, \gamma_1^u, \dots, \gamma_m^u) \cdot w_{i,j}, \quad \gamma_{ij}^{u^{-1}} = (\gamma_0^{u^{-1}}, \gamma_1^{u^{-1}}, \dots, \gamma_m^{u^{-1}}) \cdot w_{i,j}^{\vee},$$
$$H = {}^t(\gamma_0^u, \gamma_1^u \dots, \gamma_m^u) \bullet (\gamma_0^{u^{-1}}, \gamma_1^{u^{-1}} \dots, \gamma_m^{u^{-1}}),$$

where $\bullet$ is the matrix extension of the intersection form $\mathcal{I}$.
For $\gamma^u = (\gamma_0^u, \gamma_1^u, \dots, \gamma_m^u) \cdot {}^t(v_0, v_1 \dots, v_m) \in H_1(\mathbb{C}_{\dot{x}}, u)$, we have

$$\begin{aligned}
&\mathcal{I}(\gamma^u, \gamma_{ij}^{u^{-1}}) \\
=&(v_0, v_1, \dots, v_m) \cdot {}^t(\gamma_0^u, \gamma_1^u, \dots, \gamma_m^u) \bullet (\gamma_0^{u^{-1}}, \gamma_1^{u^{-1}} \dots, \gamma_m^{u^{-1}}) \cdot w_{ij}^{\vee} \\
=&(v_0, v_1, \dots, v_m) H w_{ij}^{\vee} = {}^t w_{ij}^{\vee} \, {}^t H \, {}^t(v_0, v_1, \dots, v_m), \\
&\mathcal{I}(\gamma_{ij}^u, \gamma_{ij}^{u^{-1}}) = {}^t w_{ij}^{\vee} \, {}^t H w_{ij}.
\end{aligned}$$

To get M_{ij}, substitute these into the expressions of $\mathcal{M}_{ij}$ in Theorem 4.6. □

5. Lauricella's hypergeometric systems $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_C$

Lauricella's hypergeometric systems $\mathcal{F}_A$,$\mathcal{F}_B$,$\mathcal{F}_C$ are systems of differential equations in m-variables $x_1, \dots, x_m$ of rank 2^m.

The systems $\mathcal{F}_A$, $\mathcal{F}_B$, $\mathcal{F}_C$ have parameters

$$\begin{array}{llll}
\mathcal{F}_A: & a; & b_1, \dots, b_m; & c_1, \dots, c_m; \\
\mathcal{F}_B: & a_1, \dots, a_m; & b_1, \dots, b_m; & c; \\
\mathcal{F}_C: & a; & b; & c_1, \dots, c_m.
\end{array}$$

Since $\mathcal{F}_B$ is obtained by the variable change $1/x_1, \dots, 1/x_m$ for $\mathcal{F}_A$ with parameters $(1-c+a_1+\cdots+a_m; a_1, \dots, a_m; a_1-b_1+1, \dots, a_m-b_m+1)$, the monodromy representation of $\mathcal{F}_B$ is given by that of $\mathcal{F}_A$.

The singular locus of $\mathcal{F}_A$ is

$$\Big(\prod_{i=1}^{m} x_i\Big) \cdot \Big(\prod_{i=1}^{m}(1-x_i)\Big) \cdot \Big(\prod_{1\le i<j\le m}(1-x_i-x_j)\Big)$$
$$\cdot \Big(\prod_{1\le i<j<k\le m}(1-x_i-x_j-x_k)\Big) \cdots (1-x_1-\cdots-x_m) = 0,$$

and that of $\mathcal{F}_C$ is

$$\Big(\prod_{i=1}^{m} x_i\Big) \cdot R(x_1,\dots,x_m) = 0,$$

where $R(x_1,\dots,x_m)$ is a polynomial of degree 2^{m-1} given by

$$\prod_{(v_1,\dots,v_m)\in\mathbb{F}_2^m} \big(1+(-1)^{v_1}\sqrt{x_1}+\cdots+(-1)^{v_m}\sqrt{x_m}\big).$$

If $c_1, \ldots, c_m \notin \mathbb{Z}$ then each of the systems $\mathcal{F}_A$ and $\mathcal{F}_C$ admits a fundamental system of solutions around $(0, \ldots, 0) \in \mathbb{C}^m$ in terms of hypergeometric series together with factors

$$\prod_{i=1}^{m} x_i^{(1-c_i)v_i}, \quad (v_1, \ldots, v_m) \in \mathbb{F}_2^m.$$

$\mathcal{M}_0^i$ $(i = 1, \ldots, m)$: the circuit transform of $\mathcal{F}_A$ (resp. $\mathcal{F}_C$) along a loop turning $x_i = 0$. Then we can immediately see that $\mathcal{M}_0^i$ is represented by a diagonal matrix w.r.t. this fundamental system.

To get the monodromy representation of $\mathcal{F}_A$ or $\mathcal{F}_C$, we study circuit transformations caused by loops turning other divisors.

Though there are many components of the singular locus of $\mathcal{F}_A$, it is not difficult to get generators of the fundamental group of its complement since they are hyperplanes.

In case of $\mathcal{F}_C$, it was a serious problem to prove a loop turning the divisor $R(x_1,\dots,x_m)=0$ and m loops turning $x_i=0$ generate the fundamental group of the complement of the singular locus of $\mathcal{F}_C$. Its group structure is completely determined in [GK] and [T] now.

For these studies, we consider twisted homology groups associated with Euler type integrals of solutions to $\mathcal{F}_A$ or $\mathcal{F}_C$, where the integrands are

$$\begin{aligned}
\mathcal{F}_A: &\quad (1-t_1x_1-\cdots-t_mx_m)^{-a}\prod_{i=1}^m t_i^{b_i-1}(1-t_i)^{c_i-b_i-1},\\
\mathcal{F}_C: &\quad (1-t_1-\cdots-t_m)^{c_1+\cdots+c_m-a-m}\left(1-\frac{x_1}{t_1}-\cdots-\frac{x_m}{t_m}\right)^{-b}\prod_{i=1}^m t_i^{-c_i}.
\end{aligned}$$

It turns out that "Monodromy principle for the intersection form" is valid. All of these circuit transformations can be expressed by reflections w.r.t. the intersection form. Their explicit forms, refer to [MY] and [G].

6. The Aomoto-Gelfand hypergeometric system

$M(k+1, n+2)$: the set of $(k+1) \times (n+2)$ complex matrices X.
We set

$$\ell_j(X) = \sum_{i=0}^{k} t_i x_{ij}, \quad u(t, X) = \prod_{j=0}^{n+1} \ell_j(X)^{\alpha_j},$$

where $t = (t_0, \ldots, t_k)$, $X = (x_{ij})_{0 \le i \le k,\ 0 \le j \le n+1} \in M(k+1, n+2)$,

$\alpha = (\alpha_0, \ldots, \alpha_{n+1})$ satisfies $\displaystyle\sum_{j=0}^{n+1} \alpha_j = 0$.

We define $F^\alpha(X)$ by the integral over a k-chain Δ in $\mathbb{P}^k$ (t-space) with some boundary condition:

$$F^\alpha(X) = \int \cdots \int_{\Delta} u(t, X)\varphi, \quad \varphi = \bigwedge_{j=1}^{k} d_t \log \frac{\ell_j(t, X)}{\ell_0(t, X)}.$$

The function $F^{\alpha}(X)$ satisfies

$$F^{\alpha}(g \cdot X) = F^{\alpha}(X), \quad F^{\alpha}(X \cdot h) = F^{\alpha}(X) \prod_{j=0}^{n+1} h_j^{\alpha_j},$$

for $g \in GL_{k+1}(\mathbb{C})$ and $h = \mathrm{diag}(h_0, \cdots, h_{n+1}) \in GL_{n+2}(\mathbb{C})$.

By these properties, we can regard the number of variables as

$$(k+1)(n+2) - [(k+1)^2 + (n+2) - 1] = k(n-k) = km, \quad m = n - k.$$

The function $F^{\alpha}(X)$ satisfies

$$\sum_{j=0}^{n+1} x_{p,j}\frac{\partial F^{\alpha}(x)}{\partial x_{i,j}} = 0, \qquad 0 \le i, p \le k,$$

$$\sum_{i=0}^{k} x_{i,j}\frac{\partial F^{\alpha}(x)}{\partial x_{i,j}} = \alpha_j F^{\alpha}(x), \qquad 0 \le j \le n+1.$$

$$\frac{\partial^2}{\partial x_{i,j}\partial x_{p,q}}\left(\frac{F^{\alpha}(x)}{\det x\langle J_0\rangle}\right) = \frac{\partial^2}{\partial x_{i,q}\partial x_{p,j}}\left(\frac{F^{\alpha}(x)}{\det x\langle J_0\rangle}\right),$$

where $J_0 = \{0, 1, \ldots k\}$, $x\langle J_0\rangle$ is the minor of X consisting of $0, 1, \ldots, k$-th columns.

This is called the Aomoto-Gelfand hypergeometric system, and denoted by $\mathcal{F}_{(k+1,n+2)}$.

Fact 6.1

The system $\mathcal{F}_{(k+1,n+2)}$ is of rank $\begin{pmatrix} n \\ k \end{pmatrix} = \dfrac{(k+m)!}{k!m!}$ and its singular locus is

$$S = \bigcup_{J\subset\{0,\dots,n+1\}} S\langle J\rangle, \quad S\langle J\rangle = \{X \in M(k+1,n+2) \mid X\langle J\rangle = 0\},$$

where $J = \{j_0,\dots,j_k\}$ with $0 \le j_0 < j_1 < \cdots < j_k \le n+1$.

Set $M^*(k+1,n+2) = M(k+1,n+2) - S$ and choose a base point

$$\dot{X} = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \xi_0 & \xi_1 & \cdots & \xi_n & \xi_{n+1} \\ \xi_0^2 & \xi_1^2 & \cdots & \xi_n^2 & \xi_{n+1}^2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \xi_0^k & \xi_1^k & \cdots & \xi_n^k & \xi_{n+1}^k \end{pmatrix} \in M^*(k+1,n+2),$$

where $\xi_0,\dots,\xi_{n+1} \in \mathbb{R}$, $0 < \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1}$, and they satisfy generic conditions.

For $0 \leq i \leq k$, a multi-index J and $j \in J$, let ρ_J^{ij} be a loop in $M^*(k+1, n+2)$ starting from $\dot{X}$, fixing $x_{pq} = \xi_q^p$ except x_{ij}, approaching $S\langle J \rangle|_{x_{ij}} = 0$ via the upper half space of the x_{ij}-space, turning around this point positively, and tracing back.

Terasoma T. proved the following.

Theorem 6.2

The loops ρ_J^{ij} generate $\pi_1(M^*(k+1, n+2), \dot{X})$.

Let $\mathcal{M}_{\rho_J^{ij}}$ be the circuit transform of $\mathcal{F}_{(k+1,n+2)}$ along ρ_J^{ij}.
By considering k-simplexes σ in $\mathbb{P}_X^k = \mathbb{P}^k - \cup_{j=0}^{n+1}\{\ell_j(t, X) = 0\}$ and branches of $u(t, X)$ on σ, we can define a twisted homology group $H^k(\mathbb{P}_X^k, u(t, X))$.
We have also $H^k(\mathbb{P}_X^k, u(t, X)^{-1})$ and the intersection form $\mathcal{I}$ between $H^k(\mathbb{P}_X^k, u(t, X))$ and $H^k(\mathbb{P}_X^k, u(t, X)^{-1})$.

For $J = \{j_0, \ldots, j_k\}$, let Δ_J be a k-simplex in $\mathbb{P}^k_X$ given by $k+1$ hyperplanes $\ell_{j_0}(t,X)=0, \ldots, \ell_{j_k}(t,X)=0$. Let $\Delta^u_J \in H^k(\mathbb{P}^k_X, u(t,X))$ be the pair of Δ_J and a branch of $u(t,X)$ on Δ_J.

Lemma 6.3

(1) *The integral* $\displaystyle\int_{\Delta_J} u(t,X)\varphi$ *is a solution to* $\mathcal{F}_{(k+1,n+2)}$.
(2) *It is an eigenvector of* $\mathcal{M}_{\rho^{ij}_J}$ *of eigenvalue* $\lambda_{j_0}\cdots\lambda_{j_k}$*, where* $\lambda_j = \exp(2\pi\sqrt{-1}(\alpha_j))$.
(3) *The* 1*-eigenspace of* $\mathcal{M}_{\rho^{ij}_J}$ *is of dim.* $\binom{n}{k} - 1$.

Monodromy principle for the intersection form holds, and it yields the following.

Theorem 6.4

$\mathcal{M}_{\rho_J^{ij}}$ is expressed by a reflection w.r.t $\mathcal{I}$ as

$$\begin{aligned}\mathcal{M}_{\rho_J^{ij}}(\Delta^u) &= \Delta^u - \Delta_J^u \cdot (1 - \prod_{i\in J}\lambda_i) \cdot \frac{\mathcal{I}(\Delta^u, (\Delta_J^u)^\vee)}{\mathcal{I}(\Delta_J^u, (\Delta_J^u)^\vee)} \\ &= \Delta^u - \Delta_J^u \cdot \prod_{j\in J}(1-\lambda_j) \cdot \mathcal{I}(\Delta^u, (\Delta_J^u)^\vee).\end{aligned}$$

Corollay 6.5

If we use the same fundamental system of solutions to $\mathcal{F}_{(k+1,n+2)}$ as in [MSTY], then $\mathcal{M}_{\rho_J^{ij}}$ is represented by the same matrix as in [MSTY].

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