Monodromy representations for several hypergeometric systems by the rigidity

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1. Introduction

The Gauss hypergeometric series is defined by

$$F(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}x^n, \qquad \begin{array}{l}\{x\in\mathbb{C}\mid |x|<1\}\\ c\neq 0,-1,-2,\dots,\\ (a,n)=\Gamma(a+n)/\Gamma(a).\end{array}$$

It admits an Euler type integral

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_1^{\infty} t^{b-c}(t-x)^{-b}(t-1)^{c-a}\frac{dt}{t-1},$$

$(\mathrm{Re}(c)>\mathrm{Re}(a)>0)$ and satisfies hypergeometric differential equation

$$\mathcal{F}:\ \left[x(1-x)(\frac{d}{dx})^2+\{c-(a+b+1)x\}(\frac{d}{dx})-ab\right]f(x)=0. \qquad \text{(HGDE)}$$

$\mathcal{F}$ satisfies followings:

(1) $\mathcal{F}$ has regular singular points at $x = 0, 1, \infty$, i.e., for ${}^{\forall}\dot{x} \in X = \mathbb{C} - \{0, 1\}$, ${}^{\exists}\dot{U}$: a nbd. of $\dot{x}$ s.t. the space $Sol_{\mathcal{F}}(\dot{U})$ of single valued hol. sol's to $\mathcal{F}$ on $\dot{U}$ is 2-dim.

(2) For generic parameters, $\mathcal{F}$ admits following sol's around $x = 0, 1, \infty$:

$$\begin{array}{rrr} x = 0 & x = 1 & x = \infty \\ F_{01}(x) & F_{11}(x) & (\frac{1}{x})^a F_{\infty 1}(x) \\ x^{1-c} F_{02}(x) & (1-x)^{c-a-b} F_{12}(x) & (\frac{1}{x})^b F_{\infty 2}(x) \end{array}$$

$$\begin{aligned} F_{01}(x) &= F(a, b, c; x), \\ F_{02}(x) &= F(a - c + 1, b - c + 1, 2 - c; x), \\ F_{11}(x) &= F(a, b, a + b - c + 1; 1 - x), \\ F_{12}(x) &= F(c - a, c - b, c - a - b + 1; 1 - x), \\ F_{\infty 1}(x) &= F(a, a - c + 1, a - b + 1; \tfrac{1}{x}), \\ F_{\infty 2}(x) &= F(b, b - c + 1, b - a + 1; \tfrac{1}{x}). \end{aligned}$$

Take a base point $\dot{x} \in (0,1) \subset X$. For ${}^{\forall}\rho \in \pi_1(X, \dot{x})$, by corresponding a map from $f \in Sol_{\mathcal{F}}(\dot{U})$ to the analytic continuation $\rho_*(f)$ of f along ρ, we have a homomorphism

$$\mathcal{M} : \pi_1(X, \dot{x}) \ni \rho \mapsto [\mathcal{M}(\rho) : f \mapsto \rho_*(f)] \in GL(Sol(\dot{U})),$$

which is called the monodromy representation of $\mathcal{F}$.

$\rho_0, \rho_1 \in \pi_1(X, \dot{x})$: loops turning once around $x = 0, 1$ positively, set $\rho_\infty = (\rho_0 \rho_1)^{-1}$.

M_i ($i = 0, 1, \infty$) : the representation matrix of $\mathcal{M}(\rho_i)$ w.r.t. a basis of $Sol_{\mathcal{F}}(\dot{U})$.

The Jordan normal form $[M_i]$ of M_i is called the local monodromy at $x = i$. For generic a, b, c, we have

$$[M_0] = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{e}(-c) \end{pmatrix}, \ [M_1] = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{e}(c-a-b) \end{pmatrix}, \ [M_\infty] = \begin{pmatrix} \mathbf{e}(a) & 0 \\ 0 & \mathbf{e}(b) \end{pmatrix},$$

where $\mathbf{e}(a) = e^{2\pi\sqrt{-1}a}$.

Definition 1.1

$\mathcal{E}_1$: an ordinary differential equation with regular singular points. If the monodromy representation of $\mathcal{E}_1$ is determined by the local monodromy at every singular point, then $\mathcal{E}_1$ is called rigid.

Fact 1.2 (cf. [IKSY])

HGDE $\mathcal{F}$ is rigid, if its monodromy representation is irreducible.

Fact 1.3 (cf. [BH])

The generalized hypergeometric equation ${}_p\mathcal{F}_{p-1}$ in (HGDEp) is rigid, if its monodromy representation is irreducible.

Definition 1.4 (Haraoka)

$\mathcal{E}_m$: a regular holonomic system of differential equations in m variables, $S_{\mathcal{E}_m}$: the singular locus of $\mathcal{E}_m$. If the monodromy representation of $\mathcal{E}_m$ is determined by the local monodromy of every irreducible component of $S_{\mathcal{E}_m}$, then $\mathcal{E}_m$ is called rigid.

Fact 1.5 ([HU],[HK])

Appell's hypergeometric systems $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ are rigid, if their monodromy representations are irreducible.

These facts can be proved by determining circuit transformations by their local monodromy and relations of generators of $\pi_1(\mathbb{C}^m - S_{\mathcal{E}_m}, \dot{x})$.

On the other hand, there are Euler type integrals of solutions to ${}_p\mathcal{F}_{p-1}$ or Appell-Lauricella's hypergeometric systems. We have twisted homology groups associated with them, and intersection forms on them, which are invariant under circuit transformations.

In this talk, firstly we give M_0, M_1, M_∞ for HGDE $\mathcal{F}$. We implicitly use the intersection form, which is regarded as indeterminant with an unknown h. By expressing M_1 as a reflection w.r.t.it, we solve an equation of h given by the relation $M_0 M_1 = M_\infty^{-1}$ and the local monodromy $[M_\infty]$. Though we can get the results without the intersection form, our way improves the efficiency of the proof.

Secondly, we consider the monodromy representation of ${}_3\mathcal{F}_2$ or ${}_p\mathcal{F}_{p-1}$ by the same way.

Thirdly, we give the monodromy representation of Appell's hypergeometric system $\mathcal{F}_4$, or Lauricella's hypergeometric system $\mathcal{F}_C$ by this idea.

Finally, we introduce a hypergeometric system in 2 variables of rank 9 found by these considerations.

2. Monodromy for $\mathcal{F}$

Let consider the monodromy representation of $\mathcal{F}$. Take a base point $\dot{x} \in (0,1)$. Choose a basis of $Sol_{\mathcal{F}}(\dot{U})$ as

$$\begin{pmatrix} p_1 \cdot F_{01}(x) \\ p_2 \cdot x^{1-c} F_{02}(x) \end{pmatrix}$$

where p_1, p_2 are non-zero constants, and c is assumed to be non-integral.

There are twisted cycles γ_1^u,γ_2^u s.t.

$$p_1 \cdot F_{01}(x) = \int_{\gamma_1} u(t,x) \frac{dt}{t-1}, \quad p_2 \cdot F_{02}(x) = \int_{\gamma_2} u(t,x) \frac{dt}{t-1},$$

$$u(t,x) = t^{b-c}(t-x)^{-b}(t-1)^{c-a}.$$

H: the intersection matrix w.r.t γ_1^u,γ_2^u, i.e.,

$$H = \left(\mathcal{I}(\gamma_j^u, (\gamma_k^u)^\vee)\right)_{1 \leq j,k \leq 2},$$

where $(\ )^\vee$ is an operator changing the signs of a, b, c .

Lemma 1

(i) *M_0 is $\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{e}(-c) \end{pmatrix}$ w.r.t. to this basis for any p_1, p_2.*

(ii) *H is diagonal.*

Proof. (i) It is clear by the definition of $F_{01}(x)$, $F_{02}(x)$.

(ii) For $i = 0, 1, \infty$, M_i and H satisfy

$$M_i H \, {}^t M_i^\vee = H. \tag{1}$$

Set $H = (h_{jk})$. (1) for $i = 0$ yields

$$M_0^\vee = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{e}(c) \end{pmatrix}, \quad \begin{pmatrix} h_{11} & h_{12}\mathbf{e}(c) \\ \mathbf{e}(-c)h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

For c satisfying $\mathbf{e}(c) \neq 1$, we have $h_{12} = h_{21} = 0$. □

Set $H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ by unknowns $h_1, h_2, \ (h_1 h_2 \neq 0)$.

Lemma 2

v, w: eigenvectors of M_i of eigenvalues α, β. Then

$$\alpha\beta^\vee \neq 1 \Rightarrow vH\,{}^t w^\vee = 0.$$

Proof. Note that

$$vH\,{}^t w^\vee = v(M_i H\,{}^t M_i^\vee)\,{}^t w^\vee = (vM_i)H\,{}^t(wM_i)^\vee = \alpha\beta^\vee vH\,{}^t w^\vee.$$

Thus $(1-\alpha\beta^\vee)(vH\,{}^t w^\vee) = 0$ and $(vH\,{}^t w^\vee) = 0$ by $(1-\alpha\beta^\vee) \neq 0$. $\square$

We consider M_1.
Since $F_{11}(x)$, $(1-x)^{c-a-b}F_{12}(x)$ is a fundamental system of $\mathcal{F}$ around $x = 1$, the eigenvalues of M_1 are $1, \mathbf{e}(c-a-b)$.

Lemma 3

$v = (v_1, v_2)$: an eigenvector of M_1 of eigenvalue $\mathbf{e}(c-a-b)$. If $c-a-b \notin \mathbb{Z}$, $vH\,{}^t v^\vee \neq 0$ then

$$M_1 = I_2 - (1-\mathbf{e}(c-a-b))H\,{}^t v^\vee (vH\,{}^t v^\vee)^{-1} v. \qquad (2)$$

Moreover, if the monodromy representation is irreducible then $v_1 v_2 \neq 0$.

Proof. Set $M_1' = I_2 - (1-\mathbf{e}(c-a-b))H\,{}^t v^\vee (vH\,{}^t v^\vee)^{-1} v$.

Since $vM_1' = v-(1-\mathbf{e}(c-a-b))(vH\,{}^t v^\vee)(vH\,{}^t v^\vee)^{-1} v = \mathbf{e}(c-a-b)v$, v is an eigenvector of M_1' of eigenvalue $\mathbf{e}(c-a-b)$.

By Lemma 2, an eigenvector w of M_1 of eigenvalue 1 is characterized by $wH\,{}^t v^\vee = 0$.

On the other hand, for w satisfying $wH\,{}^t v^\vee = 0$, we have $wM_1' = w-(1-\mathbf{e}(c-a-b))(wH\,{}^t v^\vee)(vH\,{}^t v^\vee)^{-1} v = w$. Thus w is an eigenvector of M_1' of eigenvalue 1.

Coincidence of eigenvalues and eigenspaces of M_1 and M_1' implies $M_1 = M_1'$.

Note that this expression of M_1 is invariant under the non-zero scalar multiple of v.

If $v_2 = 0$ then we may take $v = (1, 0)$ to express M_1. By $vH\,{}^t v^\vee = h_1$, $H\,{}^t v^\vee (vH\,{}^t v^\vee)^{-1} v$ vanish except the $(1, 1)$-entry and M_1 is diagonal. Since M_0 is diagonal too, the monodromy representation becomes reducible. We can similarly show the case $v_1 = 0$. $\square$

By choosing p_1, p_2, we can normalize v to $(1, 1)$ with keeping M_0 invariant. Since the expression of M_1 in (2) is also invariant under non-zero scalar multiple of H, we normalize H to $\mathrm{diag}(1, h)$.
Then M_1 takes the following

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1 - \mathbf{e}(c - a - b)}{1 + h} \begin{pmatrix} 1 & 1 \\ h & h \end{pmatrix}.$$

We can determine the unknown h by the eigenvalues of M_∞.

Theorem 2.1

Suppose

$$a,\ b,\ c,\ a-c,\ b-c \notin \mathbb{Z}.$$

Then H *in the expression* (2) *of* M_1 *is*

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{(\mathbf{e}(c)-\mathbf{e}(a))(\mathbf{e}(c)-\mathbf{e}(b))}{\mathbf{e}(c)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)} \end{pmatrix}.$$

M_1 *is expressed as*

$$M_1 = I_2 - \begin{pmatrix} \frac{\mathbf{e}(c)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}{\mathbf{e}(a+b+c)-\mathbf{e}(a+b)} & \frac{\mathbf{e}(c)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}{\mathbf{e}(a+b+c)-\mathbf{e}(a+b)} \\ \frac{(\mathbf{e}(c)-\mathbf{e}(a))(\mathbf{e}(c)-\mathbf{e}(b))}{\mathbf{e}(a+b+c)-\mathbf{e}(a+b)} & \frac{(\mathbf{e}(c)-\mathbf{e}(a))(\mathbf{e}(c)-\mathbf{e}(b))}{\mathbf{e}(a+b+c)-\mathbf{e}(a+b)} \end{pmatrix}.$$

Proof. By

$$M_0 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{e}(-c) \end{pmatrix} - \frac{1-\mathbf{e}(c-a-b)}{1+h} \begin{pmatrix} 1 & 1 \\ \mathbf{e}(-c)h & \mathbf{e}(-c)h \end{pmatrix}$$

$$\mathrm{tr}(M_0 M_1) = 1 + \mathbf{e}(-c) + \frac{(\mathbf{e}(c-a-b)-1)(1+\mathbf{e}(-c)h)}{1+h}.$$

Since the eigenvalues of M_∞ are $\mathbf{e}(a), \mathbf{e}(b)$, we have

$$\mathrm{tr}(M_0 M_1) = \frac{(\mathbf{e}(a+b+c)+\mathbf{e}(c))h+\mathbf{e}(a+b)+\mathbf{e}(2c)}{\mathbf{e}(a+b+c)(1+h)} = \frac{1}{\mathbf{e}(a)} + \frac{1}{\mathbf{e}(b)}.$$

This is a linear equation of h, its solution is

$$h = -\frac{(\mathbf{e}(c)-\mathbf{e}(a))(\mathbf{e}(c)-\mathbf{e}(b))}{\mathbf{e}(c)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}.$$

To get M_1, substitute this value into (2). □

3. Monodromy for ${}_3\mathcal{F}_2$

The generalized hypergeometric series ${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right)$ with parameters $a_1, a_2, a_3,\ b_1, b_2$ ($b_1, b_2 \neq 0, -1, -2, \dots$) is defined by

$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1, n)(a_2, n)(a_3, n)}{(b_1, n)(b_2, n)(1, n)} x^n \quad (|x| < 1). $$

It admits an Euler type integral, and satisfies generalized hypergeometric equation

$$ \begin{aligned} {}_3\mathcal{F}_2 : \quad & (x\frac{d}{dx} + a_1)(x\frac{d}{dx} + a_2)(x\frac{d}{dx} + a_3) f(x) \\ & = \frac{d}{dx}(x\frac{d}{dx} + b_1 - 1)(x\frac{d}{dx} + b_2 - 1) f(x). \end{aligned} \qquad \text{(HGDE3)} $$

${}_3\mathcal{F}_2$ is a third order ordinary differential equation with regular singular points at $x = 0, 1, \infty$.
Riemann's scheme is the table of the characteristic exponents of ${}_3\mathcal{F}_2$ at $x = i$ $(i = 0, 1, \infty)$, which is

$x=0$	$x=1$	$x=\infty$
0	0	a_1
$1-b_1$	1	a_2
$1-b_2$	$b_1+b_2-a_1-a_2-a_3$	a_3

Table: Riemann's scheme

There are a holomorphic solution and solutions with factors x^{1-b_k} $(k = 1, 2)$ around $x = 0$;
there are two linearly independent holomorphic solutions and a solution with factor $x^{b_1+b_2-a_1-a_2-a_3}$ around $x = 1$.

Take $\dot{x} \in (0,1)$ and its n.b.d $\dot{U}$ of $\dot{x}$. We choose a basis of the space $Sol_{{}_3\mathcal{F}_2}(\dot{U})$ of solutions to ${}_3\mathcal{F}_2$ by non-zero scalar multiples of ${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right)$ and solutions with factors x^{1-b_k} $(b_1, b_2, b_1 - b_2 \notin \mathbb{Z})$.
The circuit transform M_0 of (HGDE3) w.r.t this basis is

$$M_0 = \begin{pmatrix} 1 & & \\ & \mathbf{e}(-b_1) & \\ & & \mathbf{e}(-b_2) \end{pmatrix}.$$

We may assume that

$$M_1 = I_3 - (1-\lambda)H\,{}^t v(vH\,{}^t v)^{-1} v \tag{3}$$

where $\lambda = \mathbf{e}(b_1 + b_2 - a_1 - a_2 - a_3)$, $H = \begin{pmatrix} 1 & & \\ & h_1 & \\ & & h_2 \end{pmatrix}$ is given by unknowns h_1, h_2, $v = (1,1,1)$, and $vH\,{}^t v$ is supposed to be non-zero.

We can determine unknowns h_1, h_2 by Riemann's scheme.

Theorem 3.1

Suppose that

$$a_i,\ a_i - a_{i'},\ b_j,\ b_1 - b_2,\ a_i - b_j \notin \mathbb{Z} \quad (i, i' = 1, 2, 3;\ j = 1, 2).$$

H is determined by

$$h_1 = -\frac{(\mathbf{e}(b_2) - 1)(\mathbf{e}(a_1) - \mathbf{e}(b_1))(\mathbf{e}(a_2) - \mathbf{e}(b_1))(\mathbf{e}(a_3) - \mathbf{e}(b_1))}{\mathbf{e}(b_1)(\mathbf{e}(b_2) - \mathbf{e}(b_1))(\mathbf{e}(a_1) - 1)(\mathbf{e}(a_2) - 1)(\mathbf{e}(a_3) - 1)},$$

$$h_2 = -\frac{(\mathbf{e}(b_1) - 1)(\mathbf{e}(a_1) - \mathbf{e}(b_2))(\mathbf{e}(a_2) - \mathbf{e}(b_2))(\mathbf{e}(a_3) - \mathbf{e}(b_2))}{\mathbf{e}(b_2)(\mathbf{e}(b_1) - \mathbf{e}(b_2))(\mathbf{e}(a_1) - 1)(\mathbf{e}(a_2) - 1)(\mathbf{e}(a_3) - 1)}.$$

M_1 is given by the substitution of these values into H of (3).

Proof. $G(h_1, h_2; t)$: the eigenpolynomial of $M_0 M_1$. By $M_0 M_1 = M_\infty^{-1}$ and Riemann's scheme, the equation $G(h_1, h_2; t) = 0$ w.r.t. t should has solutions $t = \mathbf{e}(-a_1), \mathbf{e}(-a_2), \mathbf{e}(-a_3)$.

Thus we have equations

$$\begin{aligned} G(h_1, h_2, \mathbf{e}(-a_1)) &= 0, \\ G(h_1, h_2, \mathbf{e}(-a_2)) &= 0, \\ G(h_1, h_2, \mathbf{e}(-a_3)) &= 0, \end{aligned}$$

w.r.t h_1, h_2.

Since

$$\begin{aligned} \det(M_0 M_1) =& \det(M_0)\det(M_1) = \mathbf{e}(-b_1 - b_2)\mathbf{e}(b_1 + b_2 - a_1 - a_2 - a_3) \\ =& \mathbf{e}(-a_1 - a_2 - a_3) = \mathbf{e}(-a_1)\mathbf{e}(-a_2)\mathbf{e}(-a_3), \end{aligned}$$

the last equation is not independent of the first and second ones.

This system reduces to a system of linear equations, solve it. $\square$

The generalized hypergeometric series ${}_pF_{p-1}\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_{p-1}\end{matrix};x\right)$ $(b_1,\dots,b_{p-1}\neq 0,-1,-2,\dots)$ is defined by

$$\sum_{n=0}^{\infty}\frac{(a_1,n)\cdots(a_{p-1},n)(a_p,n)}{(b_1,n)\cdots(b_{p-1},n)(1,n)}x^n \quad (|x|<1).$$

It admits an Euler type integral, and satisfies

$$\begin{aligned} {}_p\mathcal{F}_{p-1}: \quad & (x\frac{d}{dx}+a_1)\cdots(x\frac{d}{dx}+a_p)f(x) \\ &= \frac{d}{dx}(x\frac{d}{dx}+b_1-1)\cdots(x\frac{d}{dx}+b_{p-1}-1)f(x). \end{aligned} \tag{HGDEp}$$

Set $M_0=\mathrm{diag}(1,\mathbf{e}(-b_1),\dots,\mathbf{e}(-b_{p-1}))$,

$$M_1 = I_p-(1-\lambda)H\,{}^tv(vH\,{}^tv)^{-1}v, \tag{4}$$

where $v=(1,\dots,1)\in\mathbb{Z}^p$, $\lambda=\mathbf{e}(b_1+\cdots+b_{p-1}-a_1-\cdots-a_p)$, and $H=\mathrm{diag}(1,h_1,\dots,h_{p-1})$ are given by unknowns $h_1,\dots,h_{p-1}$.

Consider the eigenpolynomial $G(t, h_1, \dots, h_{p-1})$ of $M_0 M_1$. It should has solutions $t = \mathbf{e}(-a_1), \dots, \mathbf{e}(-a_p)$, which yield a system of linear equations of $h_1, \dots, h_{p-1}$. By solving it, we have the expression of M_1:

Theorem 3.2

Suppose that $a_1, \dots, a_p, b_1, \dots, b_{p-1}$ are generic. Then H is determined by

$$h_j = \frac{-\prod_{1\le k\le p-1}^{k\neq j}(\mathbf{e}(b_k)-1)\prod_{k=1}^{p}(\mathbf{e}(a_k)-\mathbf{e}(b_j))}{\mathbf{e}(b_j)\prod_{1\le k\le p-1}^{k\neq j}(\mathbf{e}(b_k)-\mathbf{e}(b_j))\prod_{k=1}^{p}(\mathbf{e}(a_k)-1)} \quad (1 \le j \le p-1).$$

M_1 is given by the substitution of these values into H of (4).

4. Monodromy for Appell's $\mathcal{F}_4$

Appell's hypergeometric series F_4 is defined by

$$F_4(a,b,c;x) = \sum_{n_1,n_2=0}^{\infty} \frac{(a,n_1+n_2)(b,n_1+n_2)}{(c_1,n_1)(c_2,n_2)(1,n_1)(1,n_2)} x_1^{n_1} x_2^{n_2},$$

where $a, b, c = (c_1, c_2)$ $(c_1, c_2 \neq 0, -1, -2, \dots)$ are complex parameters.

It converges on $\{x = (x_1, x_2) \in \mathbb{C}^2 \mid \sqrt{|x_1|} + \sqrt{|x_2|} < 1\}$, admits an Euler type integral with integrand

$$t_1^{-c_1} t_2^{-c_2} (1 - t_1 - t_2)^{c_1+c_2-a-2} (1 - x_1/t_1 - x_2/t_2)^{-b},$$

and satisfies Appell's hypergeometric system $\mathcal{F}_4(a,b,c)$

$$\Big[x_1(1-x_1)\partial_1^2 - x_2^2\partial_2^2 - 2x_1x_2\partial_1\partial_2 + \{c_1 - (a+b+1)x_1\}\partial_1$$
$$-(a+b+1)x_2\partial_2 - ab\Big] f(x) = 0,$$
$$\Big[x_2(1-x_2)\partial_2^2 - x_1^2\partial_1^2 - 2x_1x_2\partial_1\partial_2 + \{c_2 - (a+b+1)x_2\}\partial_2$$
$$-(a+b+1)x_1\partial_1 - ab\Big] f(x) = 0.$$

Fact 4.1

$\mathcal{F}_4(a, b, c)$ is a regular holonomic system of rank 4 with singular locus

$$\begin{aligned} S =& \{x \in \mathbb{C}^2 \mid x_1 x_2 R(x) = 0\}, \\ R(x) =& (1 - x_1 - x_2)^2 - 4x_1x_2 \\ =& (1-\sqrt{x_1}-\sqrt{x_2})(1-\sqrt{x_1}+\sqrt{x_2})(1+\sqrt{x_1}-\sqrt{x_2})(1+\sqrt{x_1}+\sqrt{x_2}). \end{aligned}$$

Fact 4.2

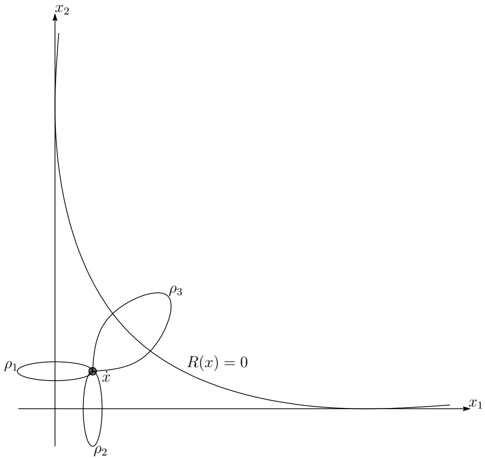
If $c_1, c_2 \notin \mathbb{Z}$, then there are 4 sol's to $\mathcal{F}_4(a, b, c)$ around $x = (0, 0)$:

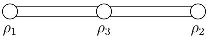
$$\begin{aligned} &F_4(a, b, c; x), \\ &x_1^{1-c_1} F_4(a + 1 - c_1, b + 1 - c_1, 2 - c_1, c_2; x), \\ &x_2^{1-c_2} F_4(a + 1 - c_2, b + 1 - c_2, c_1, 2 - c_2; x), \\ &x_1^{1-c_1} x_2^{1-c_2} F_4(a + 2 - c_1 - c_2, b + 2 - c_1 - c_2, 2 - c_1, 2 - c_2; x). \end{aligned}$$

Set $X = \mathbb{C}^2 - S$, and $\dot{x} = (\varepsilon, \varepsilon)$ for a small positive real number ε.

Fact 4.3 ([K])

$\pi_1(X, \dot{x})$ is generated by loops ρ_1, ρ_2, ρ_3. They satisfy relations $\rho_1\rho_2 = \rho_2\rho_1$, $(\rho_j\rho_3)^2 = (\rho_3\rho_j)^2$ $(j = 1, 2)$.





$\mathcal{M}$: the monodromy representation of $\mathcal{F}_4$,
$\mathbf{F}(x)$: a column vector aligned non-zero scalar multiples of sol's in Fact 4.2,
M_j ($j = 1, 2, 3$): the representation matrix $\mathcal{M}(\rho_j)$ w.r.t. $\mathbf{F}(x)$.

Lemma 4

$$M_1 = \begin{pmatrix} 1 & & & \\ & \mathbf{e}(-c_1) & & \\ & & 1 & \\ & & & \mathbf{e}(-c_1) \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \mathbf{e}(-c_2) & \\ & & & \mathbf{e}(-c_2) \end{pmatrix}.$$

Proof. It is clear by Fact 4.2. □

Lemma 5

H: the intersection matrix of the basis of twisted homology group corresponding to $\mathbf{F}(x)$. Then H is diagonal.

Proof. By $M_j H \, {}^t M_j^{\vee} = H$ ($j = 1, 2$), entries of H should be 0 except diagonal ones. □

Set $H = \mathrm{diag}(1, h_1, h_2, h_{12})$ with unknowns h_1, h_2, h_{12}.

Lemma 6

Suppose that $\mathcal{M}$ is irreducible. Then the 1-eigenspace of M_3 is 3-dim. λ (an unknown): the eigenvalue of M_3 different from 1. If the λ-eigenvector v satisfies $vH\,{}^t v^\vee \neq 0$ then

$$M_3 = I_4 - (1-\lambda)H\,{}^t v^\vee (vH\,{}^t v^\vee)^{-1} v, \tag{5}$$

and we can normalize v to $(1,1,1,1)$ by non-zero scalar multiples.

Proof. There are 3 indep. integral areas $\{(t_1, t_2) \in \mathbb{R}^2 \mid t_1, t_2 < 0\}$, $\{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 < 0, t_1 + t_2 > 1\}$, $\{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 < 0, t_1 + t_2 > 1\}$, which are invariant under the continuation along ρ_3. They span the 1-eigenspace of M_3.

This space can be expressed as $\{w \in \mathbb{C}^4 \mid wH\,{}^t v^\vee = 0\}$ by the λ-eigenvector v. Thus M_3 can be expressed as a reflection w.r.t. H.

If the j-entry of v is 0, then entries in j-th row and j-th column of M_3 become 0 except (j,j)-entry. Since M_1, M_2 are diagonal, $\mathcal{M}$ is reducible.

Thus we can normalize v to $(1,1,1,1)$ $\square$

To get M_3 explicitly, we determine the unknowns h_1, h_2, h_{12} in H and the eigenvalue λ of M_3.

Theorem 4.4

Suppose that a, b, c_1, c_2 are generic.

$$\begin{aligned} h_1 &= -\frac{(\mathbf{e}(c_1)-\mathbf{e}(a))(\mathbf{e}(c_1)-\mathbf{e}(b))}{\mathbf{e}(c_1)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}, \\ h_2 &= -\frac{(\mathbf{e}(c_2)-\mathbf{e}(a))(\mathbf{e}(c_2)-\mathbf{e}(b))}{\mathbf{e}(c_2)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}, \\ h_{12} &= \frac{(\mathbf{e}(c_1+c_2)-\mathbf{e}(a))(\mathbf{e}(c_1+c_2)-\mathbf{e}(b))}{\mathbf{e}(c_1+c_2)(\mathbf{e}(a)-1)(\mathbf{e}(b)-1)}, \\ \lambda &= -\mathbf{e}(c_1+c_2-a-b). \end{aligned}$$

M_3 *is given by the substitution of these values into* (5).

Proof. Set $x_2 = 0$, then $F_4(a, b, c_1, c_2; x_1, x_2)$ reduces to $F(a, b, c_1; x_1)$. Since this restriction corresponds to taking $1, 2$-th rows from $\mathbf{F}(x)$, we can determine h_1 by the way in §2.

Similarly we can determine h_2 by setting $x_1 = 0$.

To determine h_{12}, take $3, 4$-th rows from $\mathbf{F}(x)$ divide $x_2^{1-c_2}$, and consider the restriction to $x_2 = 0$. Since it is regarded as $\mathcal{F}(a + 1 - c_2, b + 1 - c_2, c_1)$, h_{12}/h_2 is equal to h for $\mathcal{F}(a + 1 - c_2, b + 1 - c_2, c_1)$ in §2.

By $(\rho_1\rho_3)^2 = (\rho_3\rho_1)^2$, we have $(M_1M_3)^2 = (M_3M_1)^2$, which yields a quadratic equation w.r.t. λ. Its solutions are $-\mathbf{e}(c_1 + c_2 - a - b)$ and 1. If $\lambda = 1$ then $M_3 = I_4$, and the monodromy representation is reducible. $\square$

5. Monodromy for Lauricella's $\mathcal{F}_C$

Appell's $F_4(a,b,c_1,c_2;x_1,x_2)$ is generalized to Lauricella's $F_C(a,b,c;x)$:

$$F_C(a,b,c;x)=\sum_{n_1,\dots,n_m=0}^{\infty}\frac{(a,n_1+\cdots+n_m)(b,n_1+\cdots+n_m)}{(c_1,n_1)\cdots(c_m,n_m)(1,n_1)\cdots(1,n_m)}x_1^{n_1}\cdots x_m^{n_m},$$

where $x=(x_1,\dots,x_m)$, $a,b,c=(c_1,\dots,c_m)$ $(c_1,\dots,c_m\neq 0,-1,-2,\dots)$ are complex parameters.

It converges on $\{x=(x_1,x_2)\in\mathbb{C}^2 \mid \sqrt{|x_1|}+\cdots+\sqrt{|x_m|}<1\}$, admits an Euler type integral, and satisfies Lauricella's hypergeometric system

$$\mathcal{F}_C(a,b,c):\Big[x_i(1-x_i)\partial_i^2-x_i\sum_{\substack{j\neq i\\1\le j\le m}}x_j\partial_i\partial_j-\sum_{\substack{j_1\neq i\\1\le j_1,j_2\le m}}x_{j_1}x_{j_2}\partial_{j_1}\partial_{j_2}$$
$$+\{c_i-(a+b+1)x_i\}\partial_i-(a+b+1)\sum_{\substack{j\neq i\\1\le j\le m}}x_j\partial_j-ab\Big]f(x)=0$$

for $1\le i\le m$.

$\mathcal{F}_C(a,b,c)$ is a regular holonomic system of rank 2^m with singular locus $S = \{(x_1,\dots,x_m) \in \mathbb{C}^m \mid x_1 \dots x_m R(x) = 0\}$, where

$$R(x) = \prod_{\delta_1,\dots,\delta_m=\pm 1} (1 + \delta_1\sqrt{x_1} + \dots + \delta_m\sqrt{x_m}).$$

Fact 5.1

If $c_1,\dots,c_m \notin \mathbb{Z}$, then there are 2^m sol's to $\mathcal{F}_C(a,b,c)$ around $(0,\dots,0)$:

$$\Big[\prod_{i\in I_r} x_i^{1-c_i}\Big] F_C(a + \sum_{i\in I_r}(1-c_i), b + \sum_{i\in I_r}(1-c_i), c + 2\sum_{i\in I_r}(1-c_i)e_i; x),$$

where $0 \le r \le m$, $I_r = \{i_1,\dots,i_r\} \subset \{1,\dots,m\}$, and e_i is the i-th unit row vector of $\mathbb{C}^m$.

$\mathbf{F}_C(x)$: a column vector consisting of non-zero scalar multiples of sol's in Fact 5.1, where they are aligned by an order for $I_r \subset \{1,\dots,m\}$:

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \dots, \{1,2,3\}, \{4\}, \dots, \{1,\dots,m\}.$$

Set $X=\mathbb{C}^m-S$, and $\dot{x}=(\varepsilon,\dots,\varepsilon)$ for a small positive real number ε.

ρ_i $(1\le i\le m)$: a loop in X with terminal $\dot{x}$ turning once positively around $x_i=0$,
ρ_{m+1} : that around $R(x)=0$.

Fact 5.2 ([G],[GK],[T])

(i) $\pi_1(X,\dot{x})$ is generated by loops $\rho_1,\dots,\rho_m,\rho_{m+1}$.

(ii) They satisfy

$$\rho_j\rho_k=\rho_k\rho_j,\quad (\rho_j\rho_{m+1})^2=(\rho_{m+1}\rho_j)^2,\quad (1\le j<k\le m),$$

$$(\rho_J^{-1}\rho_{m+1}\rho_J)(\rho_K^{-1}\rho_{m+1}\rho_K)=(\rho_K^{-1}\rho_{m+1}\rho_K)(\rho_J^{-1}\rho_{m+1}\rho_J),$$

where $J,K\subset\{1,\dots,m\}$ satisfying $J\neq\emptyset$, $K\neq\emptyset$, $J\cap K=\emptyset$, $\#J+\#K\le m-1$, and $\rho_J=\prod_{j\in J}\rho_j$.

(iii) The circuit transformation of ρ_{m+1} for $\mathcal{F}_C(a,b,c)$ is a reflection w.r.t. the intersection form between twisted homology groups associated with Euler type integrals.

We can see the monodromy representation $\mathcal{M}$ of $\mathcal{F}_C(a,b,c)$ by similar consideration. Suppose that $\mathcal{M}$ is irreducible.

$M_1, \dots, M_m, M_{m+1}$: the circuit transformations of $\rho_1, \dots, \rho_m, \rho_{m+1}$ for $\mathcal{F}_C(a,b,c)$ w.r.t $\mathbf{F}_C(x)$.

$M_1, \dots, M_m$ are diagonal matrices. The diagonal entry $d_j(I)$ of M_j corresponding a subset $I \subset \{1, \dots, m\}$ is

$$d_j(I) = \begin{cases} \mathbf{e}(-c_j) & \text{if} \quad j \in I, \\ 1 & \text{if} \quad j \notin I. \end{cases}$$

By setting $v = (1, \dots, 1) \in \mathbb{Z}^{2^m}$, $H = \mathrm{diag}(1, h_1, h_2, h_{12}, \dots, h_{1\dots m})$ with unknowns $h_1, h_2, h_{12}, \dots, h_{1\dots m}$, we can express M_{m+1} as a complex reflection

$$M_{m+1} = I_{2^m} - (1-\lambda) H\,{}^t v (vH\,{}^t v)^{-1} v \tag{6}$$

where λ is an unknown, and $vH\,{}^t v$ is supposed to be non-zero.

We can determine these unknowns.

Theorem 5.3

Suppose that $a, b, c_1, \ldots, c_m$ are generic.
For $I \subset \{1, \ldots, m\}$ $(I \neq \emptyset)$,

$$\begin{aligned} h_I &= (-1)^{\#(I)} \frac{(\mathbf{e}(\sum_{k\in I} c_k) - \mathbf{e}(a))(\mathbf{e}(\sum_{k\in I} c_k) - \mathbf{e}(b))}{\mathbf{e}(\sum_{k\in I} c_k)(\mathbf{e}(a) - 1)(\mathbf{e}(b) - 1)}, \\ \lambda &= (-1)^{m+1}\mathbf{e}(c_1 + \cdots + c_m - a - b). \end{aligned}$$

By substituting these values into (6), we have an expression of M_{m+1}.

Proof. By restricting $\mathbf{F}_C(x)$ to $x_j = 0$, we can determine h_I inductively. Substitute these into (6), then the expression of M_{m+1} has an unknown λ. By comparing components of $(M_1 M_{m+1})^2 = (M_{m+1} M_1)^2$ we have a quadratic equation of λ, whose solutions are 1 and $(-1)^{m+1}\mathbf{e}(c_1 + \cdots + c_m - a - b)$. Note that we cannot take $\lambda = 1$. $\square$

6. A hypergeometric system in 2 variables of rank 9

In [KMO1], [KMO2], we study a hypergeometric function in x_1, x_2:

$$F\left(\begin{matrix} a \\ B \end{matrix}; x\right) = F\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2, 1 \\ b_3, b_4, 1 \end{matrix}; x_1, x_2\right)$$

$$= \sum_{n_1, n_2 = 0}^{\infty} \frac{(a_1, n_1+n_2)(a_2, n_1+n_2)(a_3, n_1+n_2)}{(b_1, n_1)(b_2, n_1)(1, n_1)(b_3, n_2)(b_4, n_2)(1, n_2)} x_1^{n_1} x_2^{n_2},$$

which is defined by Kampé de Fériet.

This is a generalization of ${}_3F_2$ since the restriction of $F\left(\begin{matrix} a \\ B \end{matrix}; x\right)$ to $x_2 = 0$ or $x_1 = 0$ reduces to ${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x_1\right)$ or ${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_3, b_4 \end{matrix}; x_2\right)$.

Proposition 6.1

$F\left(\begin{matrix} a \\ B \end{matrix}; x\right)$ satisfies differential equations

$$\begin{aligned} & \theta_1(b_1-1+\theta_1)(b_2-1+\theta_1)f(x) \\ = \;& x_1(a_1+\theta_1+\theta_2)(a_2+\theta_1+\theta_2)(a_3+\theta_1+\theta_2)f(x), \\ & \theta_2(b_3-1+\theta_2)(b_4-1+\theta_2)f(x) \\ = \;& x_2(a_1+\theta_1+\theta_2)(a_2+\theta_1+\theta_2)(a_3+\theta_1+\theta_2)f(x), \end{aligned}$$

where $\theta_i = x_i \dfrac{\partial}{\partial x_i}$ $(i = 1, 2)$.

Proposition 6.2

The system $\mathcal{F}\left(\begin{matrix} a \\ B \end{matrix}\right)$ of differential equations in Proposition 6.1 is a regular holonomic system of rank 9 with singular locus

$$S = \{x \in \mathbb{C}^2 \mid x_1x_2R_3(x) = 0\}, \quad R_3(x) = (1-x_1-x_2)^3 - 27x_1x_2.$$

Set $X=\mathbb{C}^2-S$, and $\dot{x}=(\varepsilon,\varepsilon)$ for a small positive real number ε.

Theorem 6.3 ([KMO2][Theorem 6.1])

$\pi_1(X,\dot{x})$ *is isomorphic to*

$$\left\langle \rho_1,\rho_2,\rho_3 \;\middle|\; \begin{array}{l} \rho_1\rho_2=\rho_2\rho_1, \quad (\rho_j\rho_3)^3=(\rho_3\rho_j)^3 \ (j=1,2), \\ (\rho_1\rho_3\rho_1^{-1})(\rho_2\rho_3\rho_2^{-1})=(\rho_2\rho_3\rho_2^{-1})(\rho_1\rho_3\rho_1^{-1}) \end{array} \right\rangle.$$

We have 9 solutions to $\mathcal{F}\left(\begin{smallmatrix} a \\ B \end{smallmatrix}\right)$ around $(0,0)$ by using series with factors 1, $x_1^{1-b_1}$, $x_1^{1-b_3}$, $x_2^{1-b_2}$, $x_2^{1-b_4}$, $x_1^{1-b_1}x_2^{1-b_2}$, $x_1^{1-b_1}x_2^{1-b_4}$, $x_1^{1-b_3}x_2^{1-b_2}$, $x_1^{1-b_3}x_2^{1-b_4}$.

We have the circuit transformations along ρ_i w.r.t. this fundamental system by combining methods given in §3,4.

For details, refer to [KMO1], [KMO2].

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