Toric $K3$ hypersurfaces, hypergeometric systems and their applications to number theory

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In this talk, we will see period mappings of toric $K3$ hypersurfaces. They are controlled by differential equations. Moreover, we will see applications of the period mapping to number theory.

Contents

Please note that

- We shall omit precise proofs of results. If you have questions, please come to me after the talk. I will try to give detailed explanations.
- In section 2, the speaker will talk about results of the works [**N 2012**], [**N 2013**] and [**Hashimoto-N-Ueda, preprint**]. They appeared in several past conferences, workshops or seminar talks.
- Section 3 will be based on resent results [**N 2018**], [**N, preprint**] and [**N-Shiga, preparing**], motivated by the results of section 2.

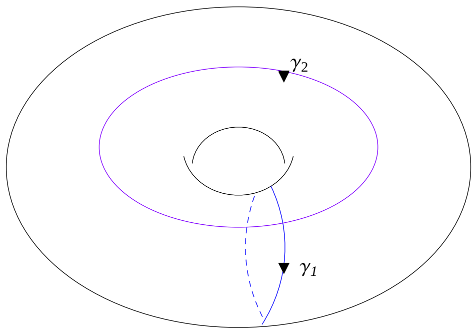
1 Periods of $K3$ surfaces

$K3$ surfaces can be regarded as a natural 2-dimensional extension of elliptic curves. We will start with elliptic curves.

1.1 Introduction: Periods of elliptic curves

An **elliptic curve** E is a compact complex curve with genus $g = 1$.

By the Riemann-Roch theorem, we can see that the canonical bundle K_E is a trivial bundle. This means that there exists the unique holomorphic 1-form ω $(\neq 0)$ on E up to a constant factor.



Let $H_1(E,\mathbb{Z})$ be the 1-homology group (= group of 1-cycles) on E. We can take 2 generators γ_1, γ_2 of $H_1(E,\mathbb{Z})$. Then,

$$\int_{\gamma_1} \omega \qquad \int_{\gamma_2} \omega$$

are called the **period integrals** on E.

If E is given by $y^2 = x(x-1)(x-\lambda)$, ω is given by $\frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$ and the period integrals are solutions of the Gauss hypergeometric equation

$${}_2E_1\Big(\frac{1}{2}, \frac{1}{2}, 1; \lambda\Big).$$

The quotient

$$\frac{\int_{\gamma_2} \omega}{\int_{\gamma_1} \omega} \in \mathbb{H}$$

is called the **period** of E.

In this talk, we will introduce $K3$ surfaces as a natural extension of elliptic curves.

1.2 Basic properties of $K3$ surfaces

Definition 1.1. *Let S be a compact complex surface. If $H^1(S, \mathcal{O}_S) = 0$ and the canonical bundle K_S is trivial, S is called a $K3$* **surface**.

A $K3$ surface is a 2 dimensional Calabi-Yau manifold. They are important in not only mathematics but also theoretical physics.

Let ω be the holomorphic 2-form. Since K_S is trivial, we can take ω uniquely up to a constant factor.

Let γ be a 2-cycle on S. The integral

$$\int_\gamma \omega$$

is called a **period integral** of S.

Let S be a $K3$ surface. The structure of the 2-homology group (= the group of 2-cycles on S) $H_2(S,\mathbb{Z})$ is well-known.

- $\mathrm{rank}(H_2(S,\mathbb{Z})) = 22$.
- $H_2(S,\mathbb{Z})$ admits a lattice structure by the canonical cup product $H_2(S,\mathbb{Z}) \times H_2(S,\mathbb{Z}) \to H_4(S,\mathbb{Z}) \simeq \mathbb{Z}$. This means that $H_2(S,\mathbb{Z})$ admits an inner product. Namely,

$$H_2(S,\mathbb{Z}) = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U.$$

where

$$E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & O & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 1 & \\ & & & & 1 & -2 & 0 & \\ & & O & & 1 & 0 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives a unimodular lattice.

$H_2(S, \mathbb{Z})$ has two important sub-lattices:

$$H_2(S, \mathbb{Z}) = \mathrm{NS}(S) \oplus \mathrm{Tr}(S).$$

- **Néron-Severi lattice**

  $\mathrm{NS}(S) = \mathrm{Div}(S)/\text{algebraic equivalence} \quad (\subset H_2(S, \mathbb{Z}))$.
  Letting ω be the holomorphic 2-form,

$$\gamma \in \mathrm{NS}(S) \Longleftrightarrow \int_\gamma \omega = 0.$$

- **Transcendental lattice**

  $\mathrm{Tr}(S)$ is the orthogonal complement of $\mathrm{NS}(S)$ with respect to the cup product of $H_2(S, \mathbb{Z})$.

$\rho = \mathrm{rank}\mathrm{NS}$ is called a **Picard number**. It is known that

$$\mathrm{NS}(S) \text{ is of type } (1, \rho - 1),$$
$$\mathrm{Tr}(S) \text{ is of type } (2, 20 - \rho).$$

Definition 1.2. *Let S be a $K3$ surface. For a lattice M, we suppose that there exists an embedding*

$$\iota : M \hookrightarrow \mathrm{NS}(S).$$

Then, the $K3$ surface S (more precisely, the pair (S, ι)) is called an M-polarized $K3$ surface.

Definition 1.3. *Let $(S_1, \iota_1), (S_2, \iota_2)$ are M-polarized $K3$ surfaces. If there exists a biholomorphic mapping $f : S_1 \to S_2$ sarisfying $\iota_1 = f^* \circ \iota_2$, (S_1, ι_1) and (S_2, ι_2) are isomorphic as M-polarized $K3$ surfaces.*

The set of isomorphism classes of M-polarized $K3$ surfaces is called the **moduli space** of M-polarized $K3$ surfaces.

Please recall that $\mathrm{rank}(H_2(S,\mathbb{Z})) = 22$. So, we have 22 period integrals. The ratio

$$\eta' = \Big(\int_{\gamma_1} \omega : \int_{\gamma_2} \omega : \cdots : \int_{\gamma_{22}} \omega \Big) \in \mathbb{P}^{21}(\mathbb{C})$$

is called the **period** of S.

In this talk, we consider the case of

$$M = \mathrm{NS}(S).$$

Since we have $\int_\gamma \omega = 0$ for $\gamma \in \mathrm{NS}(S)$, the above period η' can be reduced to a more simple form.

Let $\gamma_{r+1}, \cdots, \gamma_{22}$ be a basis of M. Then, we have the reduced period

$$\eta = \Big(\int_{\gamma_1} \omega : \cdots : \int_{\gamma_r} \omega \Big) \in \mathbb{P}^{r-1}(\mathbb{C}).$$

In fact, the reduced period

$$\eta = \left(\int_{\gamma_1} \omega : \cdots : \int_{\gamma_r} \omega \right) \in \mathbb{P}^{r-1}(\mathbb{C}).$$

satisfies the Riemann-Hodge relation

$$\eta A^t \eta = 0, \qquad \eta A \overline{^t \eta} > 0,$$

where A is the intersection matrix of the transcendental lattice $\mathrm{Tr}(S)$.

Let us consider

$$\mathcal{D} = \{\xi \in \mathbb{P}^{r-1}(\mathbb{C}) | \xi A^t \xi = 0, \xi A \overline{^t \xi} > 0\}.$$

Any period η of M-polarized $K3$ surface is an element of $\mathcal{D}$.
We note that $\mathcal{D}$ is a Hermitian symmetric space of type IV.

The stable orthogonal group

$$\tilde{O}(A) = \mathrm{Ker}(O(A) \to O(A^\vee/A)) \quad (\subset O(A)),$$

where $A^\vee = \mathrm{Hom}(A, \mathbb{Z})$, acts on $\mathcal{D}$ discontinuously.

We can consider the quotient space $\mathcal{D}/\tilde{O}(A)$.

Theorem 1.1. *(stated in* [**Dolgachev 1996**]*)*

The moduli space of pseudo-ample marked M-polarized $K3$ surfaces is given by $\mathcal{D}/\tilde{O}(A)$.

Remark 1.1. *The speaker will omit the precise definition of "pseudo-ample marked..." . This is an algebro-geometric property.*

Remark 1.2. *This theorem is essentially due to the Torelli theorem and the surjectivity of period mappings of $K3$ surfaces, by* [**Piatetski-Shapiro - Shafarevich 1971**], [**Rapoport 1977**] *and* [**Yau 1978**].

Anyway, to study the moduli of $K3$ surfaces, we need to investigate the periods of $K3$ surfaces.

2 Toric $K3$ hypersurfaces and its moduli

Background of toric $K3$ hypersurface

A **Calabi-Yau variety** is a simply-connected complex variety with the trivial canonical bundle.

2 dimensional Calabi-Yau varieties are $K3$ surfaces.

Batyrev (1994) gave a construction of Calabi-Yau varieties from Newton polytopes.

If a certain Newton polytope $P \subset \mathbb{R}^n$ is given, we can obtain $n-1$ dimensional **Calabi-Yau variety** as a divisor of n dimensional toric variety.

So, if a Newton polytope is real 3 dimensional is given, we can obtain $3-1$ dimensional Calabi-Yau varieties, namely $K3$ surfaces.

In this talk, we only consider 3 dimensional Newton polytopes,

2.1 Newton polytopes

In $\mathbb{R}^3 = \{(u, v, w)\}$, an inequality

$$au + bv + cw \leq 1, \ \ (a_j, b_j, c_j) \in \mathbb{Z}^3$$

defines a half space in $\mathbb{R}^3$.

A bounded intersection P of several half spaces gives a polytope in $\mathbb{R}^3$.

If a polytope P satisfies the conditions

(a) every vertex is a point of $\mathbb{Z}^3$,

(b) the origin is the unique inner lattice point,

(c) only the vertices are the lattice points on the boundary,

then P is called a **reflexive polytope** with at most terminal singularities.

We summerize the construction of $K3$ surfaces (= 2 dimensional Calabi-Yau varieties) from reflexive polytopes. (For details, please see the textbooks of toric varieties **[Cox], [Oda]**, $\cdots$.).

1. If a 3-dimensional reflexive polytope P is given, we can obtain the corresponding **fan** $\Delta(P)$ in $\mathbb{R}^3$ in a canonical way.

2. By a canonical argument of toric varieties, from a fan $\Delta(P)$, we can construct 3-dimensional toric variety

$$X = T_N emb(\Delta(P)).$$

   – By the general theory of toric varieties, we have

$$H^1(X, \mathcal{O}_X) = 0, \qquad H^2(X, \mathcal{O}_X(K_X)) = 0.$$

3. Then, we can easily see that anti-canonical section $S \sim -K_X$ gives a $K3$ surface as follows.

   – By the adjunction formula,

$$K_S = (K_X + S)|_S = (K_X + (-K_X))|_S = 0.$$

   – From the exact sequence

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(S, \mathcal{O}_S) \to H^2(X, \mathcal{O}_X(-S)) \to \cdots ,$$

   we have $H^1(S, \mathcal{O}_S) = 0$.

4. From such Newton polytopes, we can obtain generators of the vector space of anti-canonical sections. Letting t_1, t_2, t_3 be coordinates of X,

$$H^0(X, \mathcal{O}_X(-K_X)) = \langle t_1^a t_2^b t_3^c \rangle_{\mathbb{C}}$$

   from the lattice points $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in P \cap \mathbb{Z}^3$.

In this talk, we shall focus on the special (and interesting) case for the polytope $P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$ (columns gives the coordinates of vertices).

We have 6 lattice points $P_0 \cap \mathbb{Z}^3$:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \in \mathbb{R}^3.$$

Then, we can an anti-canonical section S on X.

$$S : c_1 t_1^0 t_2^0 t_3^0 + c_2 t_1^1 t_2^0 t_3^0 + c_3 t_1^0 t_2^1 t_3^0 + c_4 t_1^0 t_2^0 t_3^1 + c_5 t_1^0 t_2^0 t_3^{-1} + c_6 t_1^{-1} t_2^{-1} t_3^{-2} = 0,$$

where $c_1, \cdots, c_6 \in \mathbb{C}$. Namely,

$$S : c_1 + c_2 t_1 + c_3 t_2 + c_4 t_3 + c_5 t^{-1} + c_6 t_1^{-1} t_2^{-1} t_3^{-2} = 0.$$

We set

$$x = \frac{c_2 t_1}{c_1}, \quad y = \frac{c_3 t_2}{c_1}, \quad z = \frac{c_4 t_3}{c_1}, \quad \lambda = \frac{c_4 c_5}{c_1^2}, \quad \mu = \frac{c_2 c_3 c_4^2 c_6}{c_1^5}.$$

Then, S is transformed to the defining equation

$$S_0(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu = 0.$$

In this talk, we consider this defining equation.

In the following, we will see the meaning of the defining equation.

From our polytope $P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$, we set $\tilde{P}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}$.

The matrix $\tilde{P}_0$ gives a homomorphism $\mathbb{Z}^6 \to \mathbb{Z}^4$ over $\mathbb{Z}$.

Setting $L = \mathrm{Ker}(\tilde{P}_0)$, we have the exact sequence

$$0 \to L \to \mathbb{Z}^6 \xrightarrow{\tilde{P}_0} \mathbb{Z}^4 \to 0.$$

We can see that L is generated by 2 vectors

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -5 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Please note that our parameters λ, μ correspond these vectors.

$$\lambda = \frac{c_2^0 c_3^0 c_4^1 c_5^1 c_6^0}{c_1^2} = \frac{c_4 c_5}{c_1^2}, \quad \mu = \frac{c_2^1 c_3^1 c_4^2 c_5^0 c_6^1}{c_1^5} = \frac{c_2 c_3 c_4^2 c_6}{c_1^5}$$

Such a construction of parameters can be explained in the sense of **secondary stack**.

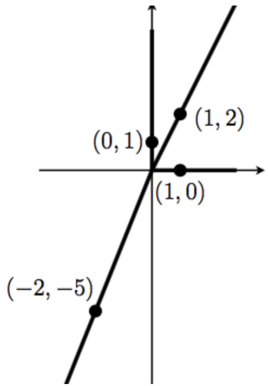
Remark 2.1. *Secondary stacks are studied by* [**Diemer-Katzarkov-Kerr 2016**] *for the purpose to study mirror symmetry of Calabi-Yau varieties. We note that secondary stacks are also very closely related to the work* [**Lafforgue 2003**].

1. By two generators of $L = \text{Ker}(\tilde{P}_0)$, we can obtain the matrix $\check{\beta} = \begin{pmatrix} -2 & 0 & 0 & 1 & 1 & 0 \\ -5 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}$. This gives a dual of the above sequence:

$$0 \to \mathbb{Z}^4 \xrightarrow{\check{\beta}} \mathbb{Z}^6 \to \check{L} \to 0.$$

   This sequence is called a **divisor sequence**.

2. From the columns of the matrix of $\check{\beta}$, we obtain a fan in $\mathbb{R}^2$. This fan is called a **secondary fan** $\mathcal{F}_{P_0}$ of the polytope P_0.



3. A pair of a fan $\mathcal{F}_{P_0}$ and the divisor sequence is called a **secondary fan** in the sense of [**Diemer-Katzarkov-Kerr 2016**]. This is a special case of **stacky fan**.

4. Generically, if a stacky fan is given, we can obtain a **toric stack**. (The construction of toric stacks are given in [**Borisov-Chena-Smith 2005**].)

5. Especially, the toric stack derived from the secondary fan and the divisor sequence is called the **Secondary stack**.

Our construction of λ, μ gives coordinates of the secondary stack $\mathcal{X}_{P_0}$. More precisely,

Theorem 2.1. *(*[**Hashimoto-N-Ueda, preprint**]*) The secondary stack $\mathcal{X}_{P_0}$ is given by a weighted blow up of weight $(1,2)$ of $\mathbb{P}(1:2:5)$ at one point. Our (λ, μ) gives the coordinates of the maximal dense torus of $\mathcal{X}_{P_0}$.*

For simplicity, we shall call (λ, μ) "a system of coordinates of the secondary stack $\mathcal{X}_{P_0}$".

2.2 Lattice structure of our $K3$ hypersurface

We will consider the moduli of toric $K3$ hypersurface

$$S_0(\lambda,\mu) : xyz^2(x+y+z+1)+\lambda xyz+\mu=0.$$

As we saw in the general theory of $K3$ surfaces, it is important to obtain the Neron-Severi lattice and the transcendental lattice.

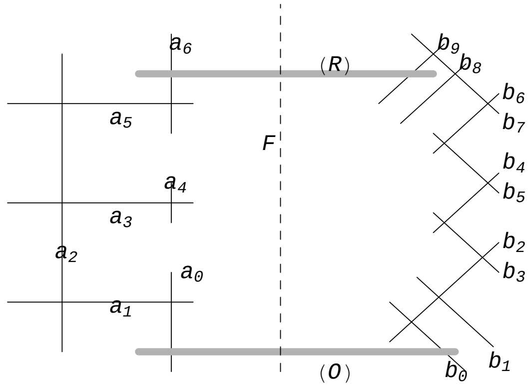
Theorem 2.2. *(***[N 2013]***) For generic* (λ,μ),

$$\begin{cases} \mathrm{NS} : E_8(-1)\oplus E_8(-1)\oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \\ \mathrm{Tr} : U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}. \end{cases}$$

Proof. (sketch)

By a birational transformation $(x,y,z)\mapsto(x_1,y_1,z_1)$, we can obtain an elliptic fibration $\pi : S_0(\lambda,\mu)\mapsto \mathbb{P}^1(\mathbb{C})$.

The singular fibres of this elliptic surface are illustrated as follows.



- By an application of the theory of **Mordell-Weil lattices** for elliptic surfaces, we can see that $\rho = \mathrm{rank}(\mathrm{NS}(S_0(\lambda,\mu))) = 18$ for generic (λ,μ).

- We can take 18 appropriate divisors from sections of π and components of singular fibres.
$\longrightarrow$ They give a basis of $\mathrm{NS}(S_0(\lambda,\mu))$.

- We can determine the structure of NS and Tr.

For detail, please see [**N 2013**] □

So, let A be the intersection matrix of the transcendental lattice:

$$A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

The moduli space of $S_0(\lambda, \mu)$ is given by $\mathcal{D}/\tilde{O}(A)$, where

$$\mathcal{D} = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{P}^4(\mathbb{C}) | \xi A^t \xi = 0, \xi A^t \overline{\xi} > 0\}.$$

The period mapping for $S_0(\lambda, \mu)$ defines a multivalued mapping

$$\mathcal{X}_{P_0} \to \mathcal{D},$$

given by

$$(\lambda, \mu) \mapsto \Big(\int_{\gamma_1} \omega : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega : \int_{\gamma_4} \omega \Big).$$

By virtue of the Torelli's theorem and the surjectivity of the period mapping, this induces a birational mapping

$$\mathcal{X}_{P_0} \dashrightarrow \mathcal{D}/\tilde{O}(A).$$

Remark 2.2. *There exist a dual polytope $\check{P}$. From $\check{P}$, we can obtain the toric variety $\check{X}$ and the corresponding $K3$ sufrace $\check{S}$ (called the mirror of S). The* **Dolgachev conjecture** *is the relation between the lattice structures of S and $\check{S}$:*

$$\mathrm{Tr}(S) = U \oplus \mathrm{NS}(\check{S}).$$

This is very important from the viewpoint of mirror symmetry of toric $K3$ hypresurfaces.

In our case, we can calculate $\mathrm{NS}(\check{S}_0)$ by a direct application of Moisezon Theorem ([**Koike 1998**]*):*

$$\mathrm{NS}(\check{S}_0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then, the Dolgachev conjecture holds for our P_0.

In [**Hashimoto-N-Ueda, preprint**]*, it is proved that the Dolgachev conjecture for reflexive polytopes for 5 vertices is true.*

2.3 Gauss-Manin Connections

The period integral $\int_\gamma \omega$ vary by the parameters λ and μ. In this subsection, we will see the behavior of period integrals.

The holomorphic 2-form of $S_0(\lambda, \mu)$ is explicitely given by

$$\omega = \frac{z dz \wedge dx}{\partial F / \partial y},$$

where $F = xyz^2(x + y + z + 1) + \lambda xyz + \mu$.

By taking an appropriate 2-cycle γ and applying the residue theorem, we have the expression of our period integrals:

$$\begin{aligned}\int_\gamma \omega &= \frac{1}{2\pi\sqrt{-1}} \iiint_\Delta \frac{z dz \wedge dx \wedge dy}{xyz^2(x + y + z + 1) + \lambda xyz + \mu} \\ &= (2\pi\sqrt{-1})^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m + 2n)!}{(m!)^3 n! (2m + n)!} \lambda^n \mu^m,\end{aligned}$$

for a 3-cycle Δ in the toric 3-fold X_{P_0}.

The triple integral $\frac{1}{2\pi\sqrt{-1}}\iiint_{\Delta}\frac{z dz\wedge dx\wedge dy}{xyz^2(x+y+z+1)+\lambda xyz+\mu}$, is a solution of the **GKZ system**

$$\begin{cases} (\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6)\eta = -\eta, \\ (\theta_2-\theta_6)\eta = 0, \quad (\theta_3-\theta_6)\eta = 0, \quad (\theta_4-\theta_5-2\theta_6)\eta = 0, \\ \dfrac{\partial^2}{\partial c_4 \partial c_5}\eta = \dfrac{\partial^2}{\partial c_1^2}\eta, \qquad \dfrac{\partial^5}{\partial c_2\partial c_3\partial c_4^2\partial c_6}\eta = \dfrac{\partial^5}{\partial^5 c_1}\eta. \end{cases}$$

from the data $\tilde{P}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}$, $\beta = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and

$$\mathrm{Ker}(\tilde{P}_0) = L = \left\langle \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

By considering the definition of λ and μ, the above system is transformed to

$$\begin{cases} D_1^{(0)} & = \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ D_2^{(0)} & = \lambda^2\theta_\mu^3 + \mu\theta_\lambda(\theta_\lambda - 1)(2\theta_\lambda + 5\theta_\mu + 1). \end{cases}$$

Our period integrals

$$\iint_\gamma \omega = \frac{1}{2\pi\sqrt{-1}} \iiint_\Delta \frac{z dz \wedge dx \wedge dy}{xyz^2(x+y+z+1) + \lambda xyz + \mu},$$

are solutions of this system.

But, the GKZ system is not enough to study the periods of our $K3$ surfaces, precisely.

- The period integrals are solutions of the above GKZ differential equation. In fact, the space of solutions of the GKZ system is 6 dimensional vector space.

- Please recall the form of our period mapping.

$$(\lambda, \mu) \mapsto \Big(\int_{\gamma_1} \omega : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega : \int_{\gamma_4} \omega \Big).$$

  We only have 4 period integrals.

This means that the GKZ equation contains 2 dimensional unnecessary solutions that are not coming from the periods of $K3$ surfaces.

We need a differential equation of rank 4 whose space of solutions is generated by 4 periods of our $K3$ surface.

Such a differential equation coincides with the **Gauss-Manin connection** for the deformation of Hodge structure of our M-polarized $K3$ surfaces $\{S_0(\lambda, \mu)\}$.

So, we need to obtain 4 dimensional subsystem of the GKZ system for periods of $K3$ surfaces.

Gauss-Manin Connection of rank 4	$\subset$	**GKZ system** of rank6
$Solution =$ Periods		$Solution =$ Periods and others

Theorem 2.3. *([**N 2012**]) Set*

$$\begin{cases} D_1^{(0)} & = \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ D_3^{(0)} & = \lambda^2(4\theta_\lambda^2 - 2\theta_\lambda\theta_\mu + 5\theta_\mu^2) - 8\lambda^3(1 + 3\theta_\lambda + 5\theta_\mu + 2\theta_\lambda^2 + 5\theta_\lambda\theta_\mu) + 25\mu\theta_\lambda(\theta_\lambda - 1). \end{cases}$$

The Gauss-Manin connection for our case is given by

$$D_1^{(0)} u = D_3^{(0)} u = 0.$$

Proof. (*sketch*) By a hard (and not sophisticated) calculation of a method of undetermined coefficients using the power series expansion

$$\int_\gamma \omega = (2\pi\sqrt{-1})^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m+2n)!}{(m!)^3 n!(2m+n)!} \lambda^n \mu^m,$$

we can directly determine the Gauss-Manin connection as a subsystem of the GKZ system (for detail, see [**N 2012**]).

□

2.4 Moduli space = Hilbert modular surface

In fact, our moduli space is closely related to a Hilbert modular surface.

Let F be a real quadratic field and let $\mathfrak{O}_F$ be the ring of integers. Let $a \mapsto a'$ be the conjugate of F over $\mathbb{Q}$.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Hilbert modular group $SL(2, \mathfrak{O}_F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$(z_1, z_2) \mapsto \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

Also, we consider the action $\tau : (z_1, z_2) \mapsto (z_2, z_1)$.

In this talk, we consider the case of $F = \mathbb{Q}(\sqrt{5})$.

Remark 2.3. $F = \mathbb{Q}(\sqrt{5})$ *is the simplest real quadratic field.*

	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{13})$	$\mathbb{Q}(\sqrt{17})$
discriminant	5	8	12	13	17

Please recall the transcendental lattice $A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, and the Hermitian symmetric space

$$\mathcal{D} = \{\eta \in \mathbb{P}^3(\mathbb{C}) | \eta A^t\eta = 0, \eta A\overline{^t\eta} > 0\} = \mathcal{D}_+ \cup \mathcal{D}_-.$$

In this case, there exists a modular isomorphism between $\mathbb{H} \times \mathbb{H}$ and $\mathcal{D}_+$:

$$\begin{array}{ccc} \mathbb{H} \times \mathbb{H} & \xrightarrow{\text{biholomorphic}} & \mathcal{D}_+ \\ \big\downarrow {\scriptstyle /\langle PSL(2,\mathfrak{O}_F),\tau\rangle} & & \big\downarrow {\scriptstyle /\tilde{O}^+(A)} \\ (\mathbb{H} \times \mathbb{H})/\langle SL(2,\mathfrak{O}_F),\tau\rangle & \xrightarrow[\text{biholomorphic}]{} & \mathcal{D}_+/\tilde{O}^+(A) = \mathcal{D}/\tilde{O}(A) \end{array}$$

- $(\mathbb{H} \times \mathbb{H})/\langle SL(2,\mathfrak{O}_F),\tau\rangle$ is called the **Hilbert modular surface**. Due to [**Hirzebruch 1977**], the Hilbert modular surface for $F = \mathbb{Q}(\sqrt{5})$ can be compactified by adding one cusp. The compactification is equal to the weighted projective space $\mathbb{P}(1:3:5)$:

$$\overline{\mathcal{D}_+/\tilde{O}^+(A)} \simeq \overline{(\mathbb{H} \times \mathbb{H})/\langle SL(2,\mathbb{Z}),\tau\rangle} \simeq \mathbb{P}(1:3:5).$$

- We have a period mapping

$$\mathcal{X}_{P_0} \xrightarrow{multivalued} \mathcal{D}_+ \xrightarrow{\text{biholomorphic}} \mathbb{H} \times \mathbb{H}$$

given by

$$(\lambda, \mu) \mapsto (z_1, z_2) = \left(-\frac{\int_{\Gamma_3} \omega + \frac{1-\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega}, -\frac{\int_{\Gamma_3} \omega + \frac{1+\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega} \right).$$

- From the Torelli theorem of $K3$ surfaces for the polarization $M = E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, we have a birational mapping

$$\mathcal{X}_{P_0} \dashrightarrow \mathbb{P}(1:3:5).$$

2.5 The secondary stack and the moduli space

In this subsection, we will obtain the explicit expression of the birational mapping

$$\begin{aligned} \mathcal{X}_{P_0} &\dashrightarrow \mathbb{P}(1:3:5) = \mathrm{Spec}(\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C}]) \\ (\lambda,\mu) &\mapsto (X,Y) = \left(\frac{\mathfrak{B}}{\mathfrak{A}^3},\frac{\mathfrak{C}}{\mathfrak{A}^5}\right) \end{aligned}$$

We will apply the theory of **holomorphic conformal structure** of partial differential equations, which were developed by Sasaki and Yoshida.

Remark 2.4. *Holomorphic conformal structures are symmetric* 2*-tensors. They are originaly studied in differential geometry.*

[**Sasaki-Yoshida 1979**] *studied them from the view point of differential equations. We apply their results to our cases.*

- The holomorphic conformal structure of the Gauss-Manin connection:

$$\Psi_{(\lambda,\mu)} = \frac{2\mu(-1+15\lambda+100\lambda^2)}{\lambda+16\lambda^2-80\lambda^3+125\mu}(d\lambda)^2 + 2(d\lambda)(d\mu) + \frac{2(\lambda^2-8\lambda^3+16\lambda^4+5\mu-50\lambda\mu)}{\mu(\lambda+16\lambda^2-80\lambda^3+125\mu)}(d\mu)^2.$$

  $\cdots$ This is calculated from the viewpoint of *partial differential equation.*

- The holomorphic conformal structure of the Hilbert modular surface $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathfrak{O}),\tau\rangle$ is calculated by **[T. Sato 1990]**:

$$\Psi_{(X,Y)} = \frac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}(dX)^2+(dX)(dY)+\frac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}(dY)^2.$$

  $\cdots$ This is calculated by the geometric structure of the orbifold $(\mathbb{H}\times\mathbb{H})/\langle SL(2,\mathfrak{O}_F),\tau\rangle$. (= *differential geometry*)

These two holomorphic conformal structure are calculated from different view points.

However, by the properties of our $K3$ surfaces, there must be a birational transformation $\varphi : (\lambda, \mu) \mapsto (X, Y)$ such that

$$\varphi^*(\Psi_{(X,Y)}) = \Psi_{(\lambda,\mu)}.$$

We can uniquely determine φ.

Theorem 2.4. *(***[N 2013], [Hashimoto-N-Ueda, preprint]***) The birational mapping φ given by*

$$(\lambda, \mu) \mapsto (1 : X : Y) = \Big(1 : \frac{25\mu}{2(\lambda - 1/4)^3} : -\frac{3125\mu^2}{(\lambda - 1/4)^5}\Big).$$

is a birational mapping

$$\mathfrak{X}_{P_0} \dashrightarrow \overline{\mathcal{D}/\tilde{O}(A)} \simeq \mathbb{P}(1 : 3 : 5).$$

(Rem) This theorem implies that

$$\mathbb{Q}(X, Y) = \mathbb{Q}(\lambda, \mu).$$

So, roughly, arithmetic properties of (λ, μ) is equal to those of (X, Y).

- The parameters (λ, μ) are coordinate of the secondary stack.

  $\to$ This is due to toric varieties and closely related to mirror symmetry of $K3$ surfaces.

- The coordinates (X, Y) are obtained from a Hilbert modular surface.

  $\to$ It concerns to **Hilbert modular functions**. They are important in number theory.

$\cdots$ Here,

Definition 2.1. *A* **Hilbert modular function** *is a meromorphic function on $\mathbb{H} \times \mathbb{H}$ which is invariant under the action of the Hilbert modular group.*

So, the above explicit theorem gives an explicit relation between the toric $K3$ surfaces and number theory.

3 Recent results

3.1 Arithmetic properties of (λ, μ)

If an elliptic curve with the period $z = \frac{\int_{\gamma_2} \omega}{\int_{\gamma_1} \omega} \in \mathbb{H}$ is given by $y^2 = 4x^3 - g_2 x - g_3$, the elliptic j-function is given by

$$\mathbb{H} \ni z \mapsto j(z) = \frac{g_2^3(z)}{g_2(z)^3 - 27 g_3^2(z)}.$$

The j-function is an elliptic modular function for $SL_2(\mathbb{Z})$.

This has a very good arithmetic property. If one takes an imaginary quadratic field K, then K defines **CM-points** z_K on $\mathbb{H}$.

Theorem (Kronecker's Jugendtraum) $K(j(z_K))/K$ is an **absolute class field** (=maximal unramified abelian extension).

- Especially,

$$\mathrm{Gal}(K(j(z_K))/K) \simeq \textbf{Ideal class group } I_K/P_K \text{ of } K.$$

Our X-function and Y-function can be regarded as a natural counterpart of j-function.

Let F be the real quadratic field for the smallest discriminant ($F = \mathbb{Q}(\sqrt{5})$). Let K be an imaginary quadratic extension of F (**CM-field**). Due to Shimura, K defines a **CM-point** $(z_{1,K}, z_{2,K}) \in \mathbb{H} \times \mathbb{H}$. We have special values $\lambda(z_{1,K}, z_{2,K}), \mu(z_{1,K}, z_{2,K})$.

Theorem (Arithmetic properties of (λ, μ), **[N 2018])**
For any CM-field K over F, $K^*(\lambda(z_{1,K}, z_{2,K}), \mu(z_{1,K}, z_{2,K}))/K^*$ *gives an* **unramified class field**.

- We note that
$$\mathbb{Q}(\lambda, \mu) = \mathbb{Q}(X, Y).$$
- K^* is the **reflex field** of K. This is also a CM-field. This is defined by the **CM-type** of K in the sense of **[Shimura, 67]**.

Proof. (sketch)

Step 1. Theta expression.

– On $\mathbb{H} \times \mathbb{H}$, there are **theta functions**, like

$$\vartheta_0(z_1, z_2)$$
$$= \sum_{m,n\in\mathbb{Z}} \exp\Big(\frac{\pi\sqrt{-1}}{2\sqrt{5}}(m,n)\begin{pmatrix}(1+\sqrt{5})z_1 - (1-\sqrt{5})z_2 & 2(z_1 - z_2)\\ 2(z_1 - z_2) & -(1-\sqrt{5})z_1 + (1+\sqrt{5})z_2\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix}\Big),$$

due to Freitag and Müller. Comparing the properties of periods and theta functions, we can obtain a theta expression of (X, Y).

Step 2. Application of the theory of Shimura varieties.

– Theta functions are often compatible with Shimura varieties. In our case, our theta expression give the **canonical model** of the **Shimura variety** (in the sense of the original work [**Shimura, 67**]) for a Hilbert modular surface.

→ Special values of X, Y generate class fields. □

■ Theta expression is a key of applications to number theory in our case.

3.2 Other families of $K3$ surfaces from the viewpoint of reflection groups

Recalling $(\lambda,\mu)\mapsto(X,Y)=(\frac{\mathfrak{B}}{\mathfrak{A}^3},\frac{\mathfrak{C}}{\mathfrak{A}^5})=(\frac{25\mu}{2(\lambda-1/4)^3}:-\frac{3125\mu^2}{(\lambda-1/4)^5})$, our toric hypresurfaces

$$xyz^2(x+y+z+1)+\lambda xyz+\mu=0.$$

is birationally equivalent to

$$z_1^2=x_1^3-4(4y_1^3-5\mathfrak{A}y_1^2)x_1^2+20\mathfrak{B}y_1^3x+\mathfrak{C}y_1^4.$$

This $K3$ surface is parametrized by

$$(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\in\mathbb{P}(1:3:5)=\mathbb{P}(2:6:10).$$

These parameters can be considered as the invariants of **icosahedral group** studied by Klein (see [**Hirzebruch 1976**] **and** [**N 2013**]).

- The icosahedral group is characterized as one of the complex reflection groups, listed by [**Shepherd-Todd, 1954**].

Shepherd-Todd		Weight	$K3$ surfaces
No. 1	real (root A_n)		
No. 2	real (root $B_n = C_n$)		
No. 3	real (root D_n)		
$\cdots$			
No. 8	complex, rank 2	$8, 12$	[**Shioda-Inose 1977**]
$\cdots$			
No. 23	complex, rank 3	$2, 6, 10$	[**N 2013**]
$\cdots$			
No. 31	complex, rank 4	$8, 12, 20, 24$	[**Clingher-Doran 2012**]
$\cdots$			
No. 33	complex, rank 5	$4, 6, 10, 12, 18$	[**N preprint**]
No. 34	complex, rank 6	$6, 12, 18, 24, 30, 42$	(The speaker is trying...)
No. 35	real, rank 6 (root E_6)		
No. 36	real, rank 7 (root E_7)		
No. 37	real, real 8 (root E_8)		

It is seems that there are good families of $K3$ surfaces attached to complex reflection groups.

On the other hand, there are good combinatrial techniques based on complex reflection groups for theta functions (by [**Runge 1993**], etc.).

- No.8
  - Shioda-Inose family

$$y^2 z - 4x^3 z + 3\alpha xz + \beta z + \frac{1}{2} = 0.$$

  - $\mathrm{wt}(\alpha, \beta) = (4, 6) = \frac{1}{2}(8, 12)$
    $\rightarrow$ **Elliptic modular forms** with **Jacobi's theta** expression.
- No.23 $\rightarrow$ **Hilbert modular forms** (as in this talk)
- No.31
  - Clingher-Doran family

$$y^2 z - 4x^3 z + 3\alpha xz + \beta z + \gamma xz^2 - \frac{1}{2}(\delta z^2 + 1) = 0.$$

  - $\mathrm{wt}(\alpha, \beta, \gamma, \delta) = (4, 6, 10, 12) = \frac{1}{2}(8, 12, 20, 24)$
    $\rightarrow$ **Siegel modular forms** with **Igusa's theta** expression

- No.33
  - **[N, preprint]** shows that the transcendental lattice A for
    $$z^2 = y^3 + (t_4 x^4 + t_{10} x^3) y + (x^7 + t_6 x^6 + t_{12} x^5 + t_{18} x^4)$$
    has a very good arithmetic property of quadratic forms, called **Kneser conditions**.
    $\rightarrow$ We can study the orbifold structure of $\mathcal{D}/\tilde{O}(A)$ effectively.
  - $(t_4, t_6, t_{10}, t_{12}, t_{18})$
    $\rightarrow$ **Hermitian modular forms** expressed by theta functions for the root lattice A_2 (**[N-Shiga, preparing]**).
- No.34 : The speaker is investigating.

The speaker is curious about such a relation among complex reflection groups, $K3$ surfaces and theta functions with applications to number theory. The speaker expects the results of No.33 and No.34 will give test cases of **complex multiplication of $K3$ surfaces** proposed by M. Schütt etc..

Thank you very much for your kind attention.