Monodromy of GKZ hypergeometric functions in terms of Mellin-Barnes integrals

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Newton polyhedron of an affine hypersurface 1

Laurent polynomial with deformation parameter coefficients $\mathbf{a} := (a_1, \cdots, a_N) \in \mathbf{T}^N = (\mathbb{C}^*)^N$,

$$F(x, x_n, \mathbf{a}) = x_n(a_1 + a_2 x^{\alpha_2} + \cdots + a_N x^{\alpha_N}). \qquad (2.1)$$

s.t. $F(x, x_n, \mathbf{a}) \in \mathbb{C}[x^{\pm}][x_n, \mathbf{a}]$ where $x^{\pm} = (x_1^{\pm}, \cdots, x_{n-1}^{\pm})$, $\{\alpha_j\}_{j=1}^N \subset \mathbb{Z}^{n-1}$.

$$\Delta(F) := \text{convex hull of } \{\alpha_j\}_{j=1}^N \subset \mathbb{R}^{n-1}.$$

$$\alpha_1 = \{0\} \in \Delta(F)^{int}. \qquad (2.2)$$

$$\bar{\Delta}(F) = \text{convex hull of } \{\bar{\alpha}_p\}_{p=1}^{N} \cup \{0\} \subset \mathbb{R}^n. \tag{2.3}$$

for

$$\bar{\alpha}_p = \begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}.$$

This $n-$dimensional polyhedron is the Newton polyhedron of

$$F(x, x_n, \mathbf{1}) + 1 = x_n f(x) + 1 \tag{2.4}$$

Associate to (2.1) a $n \times N$ matrix A,

$$\mathrm{A} = \begin{pmatrix} \bar{\alpha}_1, & \cdots, & \bar{\alpha}_N \end{pmatrix}. \tag{2.5}$$

Definition

The cone

$$\Lambda = \sum_{p=1}^{N} \mathbb{R}_{\geq 0}\bar{\alpha}_p$$

is called **Gorenstein** *if*

(1) $\sum_{p=1}^{N} \mathbb{Z}\bar{\alpha}_p = \mathbb{Z}^n$.

(2) $\exists \alpha_0^\vee \in (\mathbb{Z}^n)^\vee$ *s.t.* $< \alpha_0^\vee, \bar{\alpha}_p >= 1, \forall p \in [1, N]$.

A Gorenstein cone is called **reflexive** *if its dual cone is also Gorenstein (V.Batyrev-B.Borisov)*

$$\Lambda^\vee = \{\beta \in (\mathbb{R}^n)^\vee; < \alpha, \beta > \geq 0, \forall \alpha \in \Lambda\}.$$

Reflexive Gorenstein Cone and Reflexive Polytope

Polar polyhedron of $\Delta(F)$ (V.Batyrev)

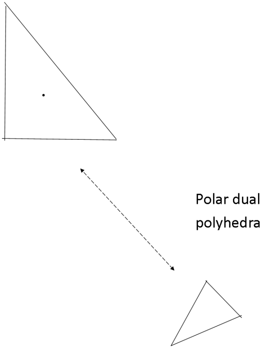
$$\Delta(F)^* := \{\beta \in (\mathbb{R}^{n-1})^\vee; <\beta, \alpha> \geq -1, \ \forall \alpha \in \Delta(F)\}.$$

If $\Delta(F)^*$ is also an integral polytope, $\Delta(F)$ is called reflexive polytope.
For $\Delta(F)$: reflexive polytope

$$(\Delta(F)^*)^* = \Delta(F).$$

$\Delta(F)$ reflexive polytope $\iff$
$\Lambda = \sum_{p=1}^{N} \mathbb{R}_{\geq 0} \bar{\alpha}_p$: reflexive Gorenstein.
$\Rightarrow$ F defines a Calabi-Yau variety.

Reflexive polyhedra



For $\bar{\Delta} = \bar{\Delta}(F)$

$$\Sigma(\bar{\Delta}) = \text{fan with rays } \{\bar{\alpha}_{\rm p}\}_{\rm p=1}^{\rm N}.$$

Toric variety $X_{\Sigma(\bar{\Delta})}$ (smooth $\iff$ $\Sigma(\bar{\Delta})$: unimodular)

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z}^n & \rightarrow & \mathbb{Z}^{\Sigma(\bar{\Delta})(1)} & \rightarrow & A_{n-1}(X_{\Sigma(\bar{\Delta})}) & \rightarrow & 0 \\ & & \mathbf{m} & \mapsto & (< \mathbf{m}, \bar{\alpha}_p >)_{p=1}^N & & & & \\ & & & & (a_p)_{p=1}^N & \mapsto & \sum_{p=1}^N a_p D_p & & \end{array}$$

with $A_{n-1}(X_{\Sigma(\bar{\Delta})})$ the Chow group (Weil divisors modulo rational equivalence) of rank $d := N - n$ of the toric variety $X_{\Sigma(\bar{\Delta})}$.

Gale transform of A (1)

Lattice $\mathbb{L} \subset \mathbb{Z}^N$ generated by $d = N - n$ integer vectors,

$$\ell_1^{(j)}\bar{\alpha}_1 + \cdots + \ell_N^{(j)}\bar{\alpha}_N = 0, \quad j \in [1; d].$$

$$\mathbb{L} = \oplus_{j=1}^{d}\mathbb{Z}\vec{\ell}^{(j)} \subset \mathbb{Z}^N, \tag{2.6}$$

where

$$B = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vdots \\ \vec{\ell}^{(d)} \end{pmatrix} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_N) \tag{2.7}$$

B: a (specially chosen) Gale transform of the $N \times n$ matrix A i.e. $\vec{\ell}^{(j)}$, $j \in [1; d]$ are orthogonal to the rows of A.

$$\vec{\ell}^{(j)} := (\ell_1^{(j)}, \cdots, \ell_N^{(j)}),\ j \in [1; d],$$

$$\mathfrak{b}_p := (\ell_p^{(1)}, \cdots, \ell_p^{(d)})^t,\ p \in [1; N].$$

Gale transform of A (2)

$$\mathrm{B} = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vdots \\ \vec{\ell}^{(d)} \end{pmatrix} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_N) \tag{2.8}$$

$$\vec{\ell}^{(j)} := (\ell_1^{(j)}, \cdots, \ell_N^{(j)}),\ j \in [1; d],$$

For every $j \in [1; d]$ define

$$\begin{aligned} I_-^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} < 0\} \\ I_+^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} > 0\} \\ I_0^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} = 0\}. \end{aligned} \tag{2.9}$$

$$I_-^{(j)} \cup I_+^{(j)} \cup I_0^{(j)} = \{1, \cdots, N\}$$

$$1 \to \mathbf{T}^n \to \mathbf{T}^N \xrightarrow{exp^{\mathrm{B}}} \mathbf{T}^d \to 1. \tag{2.10}$$

where

$$\mathrm{B} log\ \mathbf{a} = log\ \mathbf{s}$$

for $\mathbf{s} = exp^{\mathrm{B}}(\mathbf{a}) \in \mathbf{T}^d$ and $\mathbf{a} \in \mathbf{T}^N$, $N = |\Sigma(\bar{\Delta})(1)|$.
Iintroduce a deformation

$$f(x, x_n, \mathbf{s}) = x_n(\sum_{j \in \mathcal{J}} s_j x^{\alpha_j} + \sum_{\bar{j} \notin \mathcal{J}} x^{\alpha_{\bar{j}}}). \tag{2.11}$$

with $|\mathcal{J}| = d$.

Example, $n = 4$.

$$f(x, x_4, \mathbf{s}) = x_4(1 + x_1 + x_2 + \frac{s_1}{x_1x_2} + x_3 + \frac{s_2}{x_3})$$

or

$$F(x, x_4, \mathbf{a}) = x_4(a_1 + a_2x_1 + a_3x_2 + \frac{a_4}{x_1x_2} + a_5x_3 + \frac{a_6}{x_3})$$

defining the affine part of a bi-degree $(3, 2)$ K3 surface in $\mathbb{P}^2 \times \mathbb{P}^1$.

$$N = 6, d = N - n = 2.$$

$$A = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} & \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \end{array}.$$

$$\begin{array}{cc} \mathrm{B} = \begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} & \begin{array}{c} s_1 \\ s_2 \end{array} \end{array}$$

$$= \begin{pmatrix} \mathfrak{b}_1 & \mathfrak{b}_2 & \mathfrak{b}_3 & \mathfrak{b}_4 & \mathfrak{b}_5 & \mathfrak{b}_6 \end{pmatrix} = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vec{\ell}^{(2)} \end{pmatrix}$$

$$I_+^{(1)} = \{2,3,4\}, I_-^{(1)} = \{1\}, I_0^{(1)} = \{5,6\}.$$

$$I_+^{(2)} = \{5,6\}, I_-^{(1)} = \{1\}, I_0^{(1)} = \{2,3,4\}.$$

The parameter transition:

$$s_1 = \frac{a_2 a_3 a_4}{a_1^3}, \quad \frac{a_5 a_6}{a_1^2}.$$

GKZ $A-$ hypergeometric function

Residue along

$Y_{\mathbf{a}} = \{x \in \mathbf{T}^{n-1}; F(x,1,\mathbf{a}) = a_1 x^{\alpha_1} + \cdots + a_{N-1}x^{\alpha_{N-1}} + a_N = 0\},$

$$\Phi_{\gamma_{\mathbf{a}}}(\mathbf{a}) := \int_{\mathfrak{t}(\gamma_{\mathbf{a}})} F(x,1,\mathbf{a})^{-1} \frac{dx}{x^{\mathbf{1}}}, \qquad (2.12)$$

$\mathfrak{t}(\gamma_{\mathbf{a}}) \in H_{n-1}(\mathbf{T}^{n-1} \setminus Y_{\mathbf{a}})$: Leray's coboundary for $\gamma_{\mathbf{a}} \in H_{n-2}(Y_{\mathbf{a}})$ ($= S^1$ fibration over $\gamma_{\mathbf{a}}$).

Notations

$\mathbf{z} = (z_1, \cdots, z_d)$,

$\mathbf{s} = (s_1, \cdots, s_d)$,

We impose the condition : $\Delta(F)$ normal polytope i.e.

$$\mathbb{Z}^n \cap \sum_{p=1}^{N} \mathbb{R}_{\geq 0}\bar{\alpha}_p = \sum_{p=1}^{N} \mathbb{Z}_{\geq 0}\bar{\alpha}_p. \qquad (2.13)$$

Proposition

1) The GKZ $A-$HGF $\Phi_{\gamma_{\mathbf{a}}}(\mathbf{a}) \in$ Sol (A-GKZ HGS)

$$\Big(\prod_{p\in I_{+}^{(j)}} \big(\frac{\partial}{\partial a_p}\big)^{\ell_p^{(j)}} - \prod_{p\in I_{-}^{(j)}} \big(\frac{\partial}{\partial a_i}\big)^{-\ell_p^{(j)}}\Big)\Phi(\mathbf{a}) = 0, \quad j \in [1;d],$$

where $\mathbb{L} = \oplus_{j=1}^{d} \mathbb{Z}\vec{\ell}^{(j)}$ *(2.6).*

$$\sum_{p=1}^{N} \alpha_p a_p \frac{\partial}{\partial a_p}\Phi(\mathbf{a}) = 0 \ \ (weighted\ homogeneous\ of\ degree = 0)$$

$$\sum_{p=1}^{N} a_p \frac{\partial}{\partial a_p}\Phi(\mathbf{a}) = -\Phi(\mathbf{a}) \ \ (w.\ homog.\ of\ degree = -1).$$

2) dim. Sol $(A - GKZ\ HGS) = (n-1)!vol_{n-1}\Delta(F)$.

Mellin-Barnes integral representation

For $Y_{\mathbf{s}} := \{x \in \mathbf{T}^{n-1}; f(x,1,\mathbf{s}) = \sum_{j\in\mathcal{J}} s_j x^{\alpha_j} + \sum_{\bar{j}\notin\mathcal{J}} x^{\alpha_{\bar{j}}} = 0\}$, $|\mathcal{J}| = d = N - n$
period integral

$$\tilde{\Phi}_\gamma(\mathbf{s}) := \int_{\mathfrak{t}(\gamma)} f(x,1,\mathbf{s})^{-1}\frac{dx}{x^{\mathbf{1}}}, \qquad (2.14)$$

for $\mathfrak{t}(\gamma) \in H_{n-1}(\mathbf{T}^{n-1}\setminus Y_{\mathbf{s}}), \gamma \in H_{n-2}(Y_{\mathbf{s}})$.

Mellin-Barnes integral : multiple power series convergent in an open $\mathcal{V}_\rho \subset \mathbb{C}^d$:$\tilde{\Phi}_\gamma^{(\rho)}(\mathbf{s}) =$

$$\sum_{\tilde{\mathbf{z}}\in P_\rho} Res_{\mathbf{z}=\tilde{\mathbf{z}}}\Gamma(1-<\mathfrak{b}_1,\mathbf{z}>)\prod_{2\leq p\leq N}\Gamma(-<\mathfrak{b}_p,\mathbf{z}>)\varphi_\gamma(\mathbf{z})\mathbf{s}^{\mathbf{z}},$$

where $\exists\varphi_\gamma(\mathbf{z})$: periodic $\varphi_\gamma(\mathbf{z}+\mathbf{z}_0) = \varphi_\gamma(\mathbf{z})\ \forall \mathbf{z}_0 \in \mathbb{Z}^d$.

Define

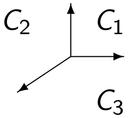
$$\Phi_{\gamma}^{(\rho)}(\mathbf{s}) = \sum_{\tilde{\mathbf{z}} \in P_{\rho}} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \prod_{1 \leq p \leq N} \Gamma(- < \mathfrak{b}_p, \mathbf{z} >) \varphi_{\gamma}(\mathbf{z}) \mathbf{s}^{\mathbf{z}}, \tag{2.15}$$

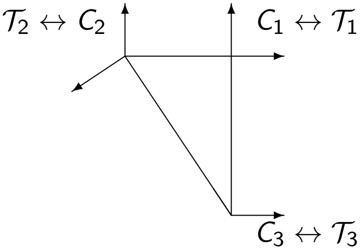
Suffix $\rho \in [1; Q]$
$\leftrightarrow \mathcal{T}_{\rho}$ regular triangulation.
$\leftrightarrow$ vertex of the secondary polytope of A
$\leftrightarrow P_{\rho}$: support of the power series $\leftrightarrow$ *cone* $-C_{\rho}^{\vee}$
$\leftrightarrow \mathcal{V}_{\rho}$ domain of convergence $\leftrightarrow$ *cone* C_{ρ} .
Here P_{ρ} :

$$- < \mathfrak{b}_p, \mathbf{z} > \in \mathbb{Z}_{\leq 0} \text{ for } p \in \mathcal{J}_{\rho} \subset [1; N] \tag{2.16}$$

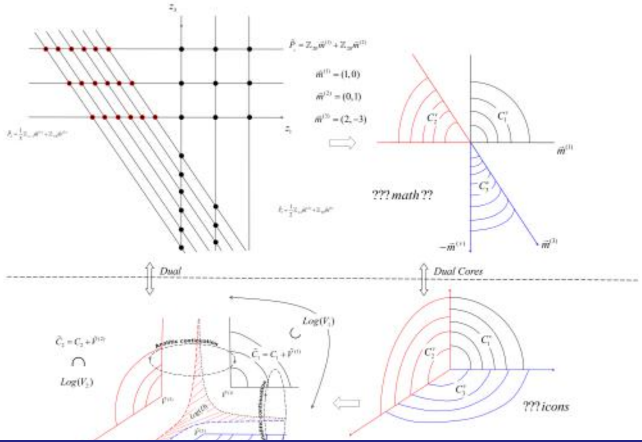
where $\mathcal{J}_{\rho} \subset [1; N]$, $|\mathcal{J}_{\rho}| = rank(\mathfrak{b}_p)_{p \in \mathcal{J}_{\rho}} = d$ $s.t. C_{\rho} = \sum_{p \in \mathcal{J}_{\rho}} \mathbb{R}_{\geq 0} \mathfrak{b}_p$. If C_{ρ} simplicial.

Secondary polytope, secondary fan





Cones associated to secondary fan



$\Phi_{\gamma}^{(\rho)}(\mathbf{s})$ satisfies HG system of Horn type,

$$\left(\prod_{p\in I_{+}^{(j)}}(-<\mathfrak{b}_p,\boldsymbol{\theta}_s>)_{\ell_p^{(j)}} - s_j\prod_{p\in I_{-}^{(j)}}(-<\mathfrak{b}_p,\boldsymbol{\theta}_s>)_{-\ell_p^{(j)}}\right)\Phi_{\gamma}^{(\rho)}$$

$= 0, \quad \forall j \in [1;d]$, where

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1),$$

the Pochhammer symbol.

$$\boldsymbol{\theta}_s = (s_1\frac{\partial}{\partial s_1},\cdots,s_d\frac{\partial}{\partial s_d}):$$

Differential operators on the algebraic torus.

Example, n=4. Continuation

Notation $e(z) = e^{2\pi i z}$.

$$\Phi_\gamma(\mathbf{s}) = \sum_{\tilde{\mathbf{z}} \in P_1} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \Gamma(3z_1+2z_2)\Gamma(-z_1)^3\Gamma(-z_2)^2\varphi_\gamma(\mathbf{z})\mathbf{s}^{\mathbf{z}} dz.$$

e.g.

$$\varphi_{\gamma_0}(z) = \left(\frac{1-e(z_1)}{2\pi i}\right)^2 \left(\frac{1-e(z_2)}{2\pi i}\right)$$

s.t.

$\Phi_{\gamma_0}(\mathbf{s}) =$

$\sum_{\tilde{\mathbf{z}} \in P_1} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \frac{\Gamma(3z_1+2z_2)\Gamma(-z_1)\Gamma(-z_2)}{\Gamma(z_1+1)^2\Gamma(z_2+1)} (e^{2\pi i}s_1)^{z_1}(e^{\pi i}s_2)^{z_2} dz_1 dz_2$

holomorphic near $(s_1, s_2) = (0,0)$ for $P_1 = (\mathbb{Z}_{\geq 0})^2$.

Dimension Sol (GKZ A-HGS) $= 6 = 3! vol(\Delta(F))$.

Definition

(Stanley-Reisner ring) Convex polyhedron $\bar{\Delta} \subset \mathbb{R}^n$ *:convex hull of*

$$\mathrm{A} = \begin{pmatrix} \bar{\alpha}_1 & \cdots & \bar{\alpha}_N \end{pmatrix}$$

triangulation $\mathcal{T}$ *of* $\bar{\Delta}$ *define the Stanley-Reisner ring for* $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_N)$,

$$\mathcal{R}_{\mathrm{A},\mathcal{T}} := \mathbb{Z}[\boldsymbol{\mu}]/(\mathcal{I}_{lin} + \mathcal{I}_{mon}), \tag{3.1}$$

- $\mathcal{I}_{lin} = \left\langle \sum_{i=1}^{N} < u^\vee, \bar{\alpha}_i > \mu_i \right\rangle, \;\; \forall u^\vee \in (\mathbb{Z}^n)^\vee.$
- $\mathcal{I}_{mon} = \langle \mu_{i_1} \cdot \mu_{i_2} \cdots \mu_{i_s} \rangle$ *for* *convex hull*$\{\bar{\alpha}_{i_1}, \cdots, \bar{\alpha}_{i_s}\}$ *not a simplex in* $\mathcal{T}$.

$\mathbb{Q}[\boldsymbol{\mu}]/\mathcal{I}_{lin} \cong \mathbb{Q}[\boldsymbol{\lambda}]$ with $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_d)$ in such a way that

$$\mathcal{R}_{A,\mathcal{T}} \otimes \mathbb{Q} \cong \mathbb{Q}[\boldsymbol{\lambda}]/\tilde{\mathcal{I}}_{mon} \tag{3.2}$$

The ideal $\tilde{\mathcal{I}}_{mon}$ in (3.2) can be written as

$$\tilde{\mathcal{I}}_{mon} = \left\langle \prod_{p \in I_+^{(1)}} < \mathfrak{b}_p, \boldsymbol{\lambda} >, \cdots, \prod_{p \in I_+^{(d)}} < \mathfrak{b}_p, \boldsymbol{\lambda} > \right\rangle . \tag{3.3}$$

for $I_+^{(j)} = \{p \in [1; N]; \ell_p^{(j)} > 0\}$.

Definition

The ideal $\mathcal{I}_{core}$ of $\mathbb{Z}[\boldsymbol{\mu}]$ is defined as a principal ideal generated by a monomial

$$\mu_{core} := \prod_{p \in \cap_j I_-^{(j)}} \mu_p.$$

Define

$$\bar{\mathcal{R}}_{A,\mathcal{T}} := \mathcal{R}_{A,\mathcal{T}} / Ann(\mathcal{I}_{core}).$$

In view of (3.2)

$$\bar{\mathcal{R}}_{A,\mathcal{T}} \otimes \mathbb{Q} \cong \tilde{\lambda}_{core} \cdot \mathbb{Q}[\boldsymbol{\lambda}] / \tilde{\mathcal{I}}_{mon}$$

for $\tilde{\lambda}_{core} = \prod_{p \in \cap_j I_-^{(j)}} < \mathfrak{b}_p, \boldsymbol{\lambda} >$.

Basis of GKZ $A-$ HG solutions

Theorem

(J.Stienstra)
(1) The cone Λ defined by the matrix A *be Gorenstein.*
$\Delta(F)$: *normal polytope.*

$\exists iso : Hom(\mathcal{R}_{\mathrm{A},\mathcal{T}}, \mathbb{C}) \cong sol\ (\ GKZ\ A-\ HGS)\ with$
$dimension = (n-1)! vol_{n-1}\Delta(F)$.

$$\exists inj : Hom(\bar{\mathcal{R}}_{\mathrm{A},\mathcal{T}}, \mathbb{C}) \hookrightarrow \mathrm{sol}\ (\mathrm{GKZ}\ A-\ \mathrm{HGS}).$$

$$\bar{\mathcal{R}}_{\mathrm{A},\mathcal{T}} := \mathcal{R}_{\mathrm{A},\mathcal{T}}/Ann(\prod_{p\in\cap_j I_-^{(j)}} < \mathfrak{b}_p, \boldsymbol{\lambda} >).$$

(2) If the the cone Λ is **reflexive Gorenstein**(+ natural conditions on $\tilde{\lambda}_{core}$), we have

$$\mathcal{R}_{A,\mathcal{T}} \otimes \mathbb{C} \cong H^*(X_{\Sigma(\Delta)}, \mathbb{C}),$$

with $X_{\Sigma(\Delta)}$: smooth projective toric variety.

$$\bar{\mathcal{R}}_{A,\mathcal{T}} = \mathcal{R}_{A,\mathcal{T}}/Ann(\mathcal{I}_{core}) \cong H^*_{toric}(W, \mathbb{Z})$$
$$:= image\ (H^*(X_{\Sigma(\Delta)}, \mathbb{Z}) \to H^*(W, \mathbb{Z})),$$

where W : a Calabi-Yau hypersurface in $X_{\Sigma(\Delta)}$ constructed by the polar polyhedron $\Delta(F)^*$.

$$\bar{\mathcal{R}}_{A,\mathcal{T}} \otimes \mathcal{O} \cong W_n H^n(\mathbf{T}^n; \{x_n f(x, 1, \mathbf{s}) + 1 = 0\}) \otimes \mathcal{O}$$

$$\cong W_{n-1} H^{n-1}(\mathbf{T}^{n-1}; \{f(x, 1, \mathbf{s}) = 0\}) \otimes \mathcal{O}$$

Example $n = 4$ continuation

$$\mathcal{R}_{A,\mathcal{T}} = \mathbb{Z}[\mu]/(\mathcal{I}_{lin} + \mathcal{I}_{mon}) \cong \mathbb{Z}[\mu_4, \mu_6]/\left\langle \mu_4^3, \mu_6^2 \right\rangle$$

$\cong \sum_{(j,k)\in[0,2]\times[0,1]} \mathbb{Z}{\lambda_1}^j {\lambda_2}^k \cong H^*(\mathbb{P}^2 \times \mathbb{P}^1)$,
$rank = 6 = 3!vol(\Delta(F))$.

$$\mathcal{I}_{lin} = \left\langle \sum_{p=1}^{6} \mu_p, \mu_2 - \mu_4, \mu_3 - \mu_4, \mu_5 - \mu_6 \right\rangle \textit{ see } \mathrm{A}$$

$$\mathcal{I}_{mon} = \langle \mu_2\mu_3\mu_4, \mu_5\mu_6 \rangle \textit{ see } \mathrm{B}$$

$\bar{\mathcal{R}}_{A,\mathcal{T}} \cong \sum_{(j,k)\in[0,2]\times[0,1]} \mathbb{Z}{\lambda_1}^j {\lambda_2}^k / Ann(-3\lambda_1 - 2\lambda_2)$
$\cong \mathbb{Z} \oplus \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \oplus \mathbb{Z}\lambda_1\lambda_2 \cong H^*_{toric}(W, \mathbb{Z})$.
$Ann(-3\lambda_1 - 2\lambda_2) = \left\langle \lambda_1^2\lambda_2, 3\lambda_1^2 - 2\lambda_1\lambda_2 \right\rangle$.
W : generic bi-degree $(3, 2)$ K3 surface in $\mathbb{P}^2 \times \mathbb{P}^1$.

Singular loci of GKZ A-HGF

Discriminantal loci $D \subset \mathbb{C}^d$ of the family of varieties $Y_{\mathbf{s}} := \{x \in \mathbf{T}^{n-1}; f(x, 1, \mathbf{s}) = 0\}$.

$$s \in D \iff Y_{\mathbf{s}} : \text{singular}.$$

Amoeba $\mathrm{Log}(D)$: $\mathrm{Log}(D \cap (\mathbb{C}^*)^d)$ by $\mathrm{Log}\colon (s_1, \dots, s_d) \mapsto (\log |s_1|, \dots, \log |s_d|)$.

Disjoint components $M_\rho, \rho \in [1, Q]$

$$\cup_{\rho=1}^{Q} M_\rho = \mathbb{R}^d \setminus \mathrm{Log}(D)$$

$Q :=$ number of vertices of the "secondary polytope" $(=$ the reduced defining equation of $D)$ of A.

$$\mathcal{V}_\rho := \mathrm{Log}^{-1}(M_\rho) \subset \mathbb{C}^d \setminus D.$$

Proposition

(GKZ,Passare-Sadykov-Tsikh, Borisov-Horja)
$\forall M_\rho \subset \mathbb{R}^d \setminus \mathrm{Log}(D)$, $\rho \in [1;Q]$, $\exists \tilde{v}^{(\rho)} \in \mathbb{R}^d$ *such that*

$$C_\rho + \tilde{v}^{(\rho)} \subset M_\rho.$$

Convex hull (P_ρ) *in* $\mathbb{R}^d = -C_\rho^\vee$.

$C_\rho^\vee := \{w \in \mathbb{R}^d; < w, v > \geq 0, \forall v \in C_\rho\}$.
$\Phi_\rho(\mathbf{s}) \in \mathcal{O}_{\mathcal{V}_\rho}$ *for all* γ *i.e.* $\forall \varphi_\gamma(\mathbf{z})$
$(\varphi_\gamma(\mathbf{z}+\mathbf{z}_0) = \varphi_\gamma(\mathbf{z}), \forall \mathbf{z}_0 \in \mathbb{Z}^d)$.

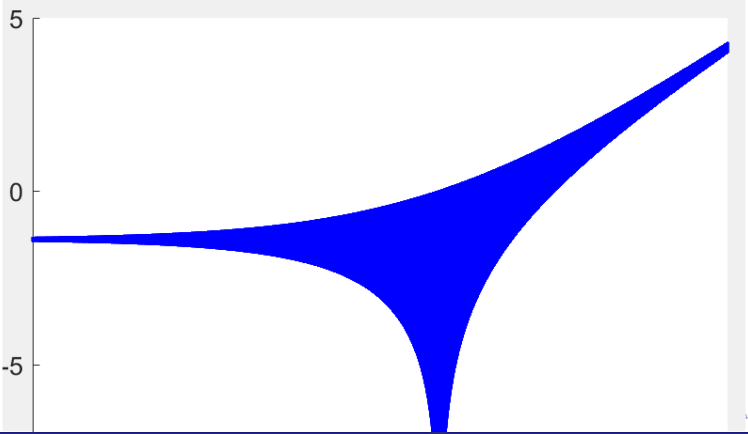
$$\Phi_\rho(\mathbf{s}) = \sum_{\tilde{\mathbf{z}} \in P_\rho} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \prod_{1 \le p \le N} \Gamma(- < \mathfrak{b}_p, \mathbf{z} >) \varphi_\gamma(\mathbf{z}) \mathbf{s}^{\mathbf{z}}, \tag{4.1}$$

Suffix $\rho \in [1; Q]$
$\leftrightarrow$ regular triangulation $\mathcal{T}_\rho$
$\leftrightarrow$ vertex of the secondary polytope of A
$\leftrightarrow$ P_ρ : support of the power series $\leftrightarrow$ *cone* $- C_\rho^\vee$
$\leftrightarrow$ $\mathcal{V}_\rho$ domain of convergence $\leftrightarrow$ *cone* C_ρ .
Here P_ρ :

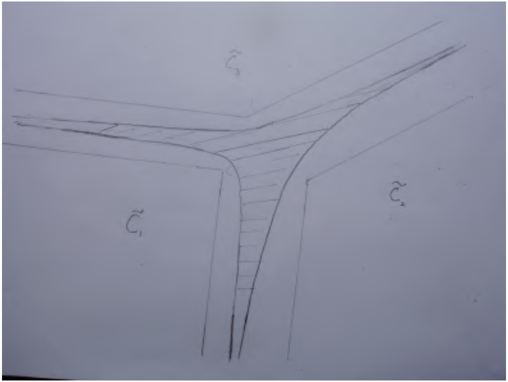
$$- < \mathfrak{b}_p, \mathbf{z} > \in \mathbb{Z}_{\le 0} \text{ for } p \in \mathcal{J}_\rho \subset [1; N] \tag{4.2}$$

where $\mathcal{J}_\rho \subset [1; N]$, $|\mathcal{J}_\rho| = rank(\mathfrak{b}_p)_{p \in \mathcal{J}_\rho} = d$.
$(\mathfrak{b}_p)_{p \in \mathcal{J}_\rho}$: generators of the cone C_ρ. If C_ρ simplicial.

Amoeba of $s_1 = (\frac{z}{3z+2})^3, s_2 = (\frac{1}{3z+2})^2$: D for Example $n = 4$.



Amoeba (D for Example $n = 4$.) and its recession cones



$$\tilde{C}_\rho = C_\rho + \tilde{v}^{(\rho)}$$

GKZ A-HG Series

Λ : Gorenstein cone. $\Delta(F)$: normal polytope.
Consider a $\mathcal{R}_{\mathrm{A},\mathcal{T},\mathbb{C}} = \mathcal{R}_{\mathrm{A},\mathcal{T}} \otimes \mathbb{C}$ (or $\bar{\mathcal{R}}_{\mathrm{A},\mathcal{T},\mathbb{C}} := \bar{\mathcal{R}}_{\mathrm{A}} \otimes \mathbb{C}$) valued solution to GKZ (or equivalently to Horn HG system)

$$\Phi_1(\mathbf{s}, \boldsymbol{\lambda}) := \sum_{\mathbf{m}\in(\mathbb{Z}_{\geq 0})^d} \varpi(\mathbf{m}+\boldsymbol{\lambda})\mathbf{s}^{\mathbf{m}+\boldsymbol{\lambda}} \quad in\ \mathcal{R}_{\mathrm{A},\mathcal{T}} \otimes \mathcal{O}_{\mathcal{V}_1} \tag{4.3}$$

with

$$\varpi(z) = \frac{1}{\prod_{p=1}^{N} \Gamma(<\mathfrak{b}_p, z> +1)}. \tag{4.4}$$

For
$Pol(\mathbf{z}') := \prod_{j=2}^{d} \frac{2\pi i}{e(z_j)-1}$

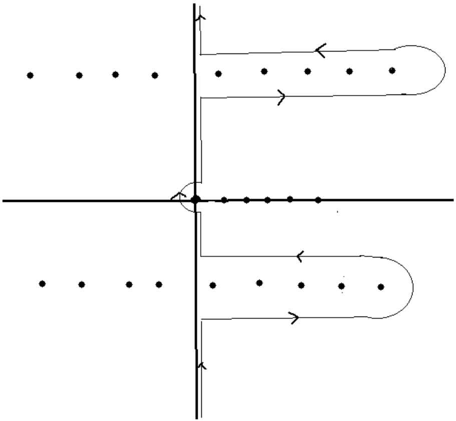
$$\Phi_1(\mathbf{s},\boldsymbol{\lambda}) = Res_{\mathbf{z}\in(\mathbb{Z}_{\geq 0})^d}\varpi(\mathbf{z}+\lambda)\frac{2\pi i}{e(z_1)-1}Pol(\mathbf{z}')\mathbf{s}^{\mathbf{z}+\boldsymbol{\lambda}}$$

$$= \int_{(-\epsilon+i\mathbb{R})^d} \frac{\mathbf{s}^{\mathbf{z}+\boldsymbol{\lambda}}}{\prod_{p=1}^{N}\Gamma(<\mathfrak{b}_p,\mathbf{z}+\boldsymbol{\lambda}>+1)} \prod_{j=1}^{d}\left(\frac{1}{1-e(z_j)}\right)d\mathbf{z}$$

$$= \sum_{\mathbf{m}\in(\mathbb{Z}_{\geq 0})^d} \varpi(\mathbf{m}+\boldsymbol{\lambda})\mathbf{s}^{\mathbf{m}+\boldsymbol{\lambda}}$$

$$\sum_{\mathbf{m}\in(\mathbb{Z}_{\geq 0})^d}\left(\frac{\prod_{p\in I_-^{(1)}}\pi^{-1}\sin(-\pi<\mathfrak{b}_p,\mathbf{m}+\boldsymbol{\lambda}>)\Gamma(-<\mathfrak{b}_p,\mathbf{m}+\boldsymbol{\lambda}>)}{\prod_{q\not\in I_-^{(1)}}\Gamma(<\mathfrak{b}_q,\mathbf{m}+\boldsymbol{\lambda}>+1)}\right)\mathbf{s}^{\mathbf{m}+\boldsymbol{\lambda}}.$$

Mellin-Barnmes contour throw



Groupoid Lemma

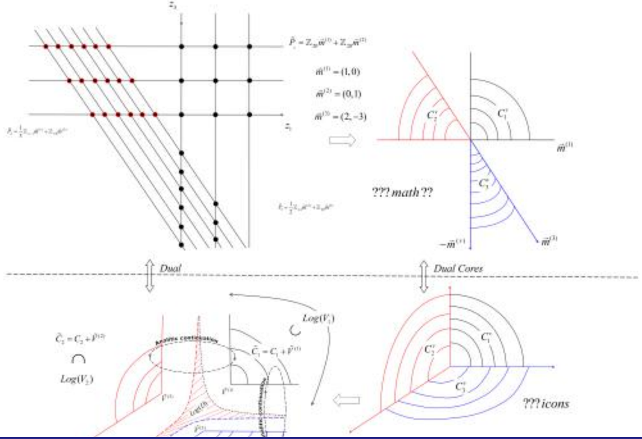
Now we consider the analytic continuation of the series $\Phi_1(\mathbf{s},\boldsymbol{\lambda})$ convergent in $\mathcal{V}_1$ (near $s_1=0$) to $\mathcal{V}_2$ (near $s_1=\infty$.) Here $e(z)=e^{2\pi i z}$.

$$\Phi_2(\mathbf{s},\boldsymbol{\lambda}) =$$
$$Res^{-}_{z_1} Res_{(\mathbb{Z}_{\geq 0})^{d-1}} \frac{2\pi i}{1-e(z_1-\lambda_1)} \prod_{p\in I_-^{(1)}} \frac{1-e(<\mathfrak{b}_p,\boldsymbol{\lambda}>)}{1-e(<\mathfrak{b}_p,(z_1,\boldsymbol{\lambda}')>)}$$
$$Pol(\mathbf{z}')\varpi(z_1,\mathbf{z}'+\boldsymbol{\lambda}')s_1^{z_1}\mathbf{s}'^{\mathbf{z}'+\boldsymbol{\lambda}'}.$$

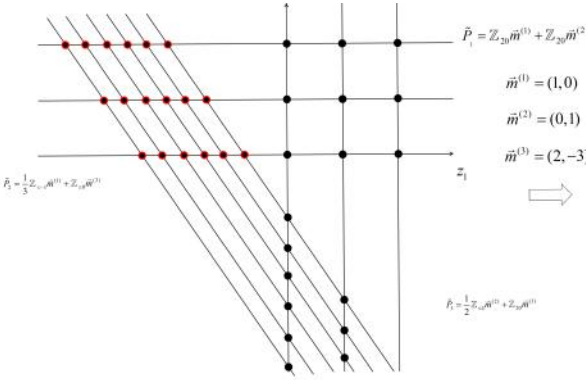
$Res^{-}_{z_1}$ means the summation of residues along

$B_-^{(1)}(\mathbf{m}'+\boldsymbol{\lambda}')=\cup_{p\in I_-^{(1)}}\{z_1;<\mathfrak{b}_p,(z_1,\mathbf{m}'+\boldsymbol{\lambda}')>\in\mathbb{Z}_{\geq 0}\}$
for $\mathbf{m}'\in(\mathbb{Z}_{\geq 0})^{d-1}$.

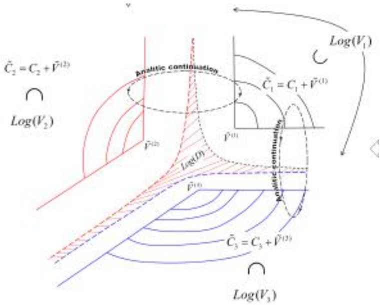
Cones associated to secondary fan



Supports P_ρ of the HG series $\Phi_\rho(\mathbf{s}, \boldsymbol{\lambda})$.



Amoebas of the discriminantal loci



The result of the analytic continuation process $\Phi_1(\mathbf{s}, \boldsymbol{\lambda})$
$\mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{V}_1$

$$\left(Res^{-}_{z_1} \frac{2\pi i}{e(z_1 - \lambda_1) - 1} \prod_{p \in I^{(1)}_{-}} \frac{1 - e(< \mathfrak{b}_p, \boldsymbol{\lambda} >)}{1 - e(< \mathfrak{b}_p, (z_1, \boldsymbol{\lambda}') >)} \right)$$

$$\cdot \left(Res^{+}_{\zeta_1} Res_{(\mathbb{Z}_{\geq 0})^{d-1}} Pol(\mathbf{z}') \prod_{q \in I^{(1)}_{+}} \frac{1 - e(- < \mathfrak{b}_q, (z_1, \boldsymbol{\lambda}') >)}{1 - e(- < \mathfrak{b}_q, (\zeta_1, \boldsymbol{\lambda}') >)} \right) \cdot$$

$$\frac{2\pi i}{e(\zeta_1 - z_1) - 1} \varpi(\zeta_1, \mathbf{z}' + \boldsymbol{\lambda}') s_1^{\zeta_1} \mathbf{s}'^{\mathbf{z}'}.$$

$Res^{+}_{\zeta_1}$: for $\mathbf{m}' \in (\mathbb{Z}_{\geq 0})^{d-1}$ sum of residues along
$B^{(1)}_{+}(\mathbf{m}' + \boldsymbol{\lambda}') = \cup_{q \in I^{(1)}_{+}} \{\zeta_1; < \mathfrak{b}_q, (\zeta_1, \mathbf{m}' + \boldsymbol{\lambda}') > \in \mathbb{Z}_{\geq 0}\}.$

Monodromy effect of a clockwise turns around $s_1 = 0, \infty$:

$$\left(Res^{-}_{z_1} \frac{2\pi i e(z_1)}{e(z_1 - \lambda_1) - 1} \prod_{p \in I^{(1)}_{-}} \frac{1 - e(< \mathfrak{b}_p, \boldsymbol{\lambda} >)}{1 - e(< \mathfrak{b}_p, (z_1, \boldsymbol{\lambda}') >)} \right)$$

$$\cdot \left(Res^{+}_{\zeta_1} Res_{(\mathbb{Z}_{\geq 0})^{d-1}} Pol(\mathbf{z}') \prod_{q \in I^{(1)}_{+}} \frac{1 - e(- < \mathfrak{b}_q, (z_1, \boldsymbol{\lambda}') >)}{1 - e(- < \mathfrak{b}_q, (\zeta_1, \boldsymbol{\lambda}') >)} \right) .$$

$$e(-\lambda_1) \frac{2\pi i}{e(\zeta_1 - z_1) - 1} \varpi(\zeta_1, \mathbf{z}' + \boldsymbol{\lambda}') s_1^{\zeta_1} \mathbf{s}'^{\mathbf{z}'}.$$

Monodromy along a loop avoiding the discriminantal loci in $s_1 \neq 0, \infty$ in clockwise way;

Monodromy theorem

Theorem

(P.R.Horja, 1999) $\bar{Y}_s$ *: Calabi-Yau hypersurface defined by a reflexive polytope* $\Delta(F)$*.*
The monodromy of $\Phi_1(\mathbf{s})$ *along a loop avoiding* $s_1 = 0, \infty$ *in a clockwise way;*

$$\Phi_1(\mathbf{s}, \boldsymbol{\lambda}) \to \Phi_1(\mathbf{s}, \boldsymbol{\lambda}) - 2\pi i \prod_{p \in I_-^{(1)}} (1 - e(< \mathfrak{b}_p, \boldsymbol{\lambda} >))$$

$$Res^+_{\zeta_1} Res_{(\mathbb{Z}_{\geq 0})^{d-1}} \left(\frac{\varpi(\zeta_1, \mathbf{z}' + \boldsymbol{\lambda}') Pol(\mathbf{z}') s_1^{\zeta_1} \mathbf{s}'^{\mathbf{z}'}}{\prod_{q \in I_+^{(1)}} 1 - e(- < \mathfrak{b}_q, (\zeta_1, \boldsymbol{\lambda}') >)} \right).$$

Here $e(z) = e^{2\pi i z}$.

Example: Generalization of the mirror Quintic by Candelas et al.

Consider $Y_s : x_1 + \cdots + x_n - \frac{s}{x_1 \cdots x_n} + 1 = 0$.
$\mathrm{B} = (1, \cdots, 1, -(n+1), 1)$.

$$\mathcal{R}_{\mathrm{A},\mathcal{T},\mathbb{C}} \cong \mathbb{C}[\lambda]/ < \lambda^{n+1} > \cong H^*(\mathbb{P}^n, \mathbb{C})$$

$\bar{\mathcal{R}}_{\mathrm{A},\mathcal{T},\mathbb{C}} \cong \mathcal{R}_{\mathrm{A},\mathbb{C}}/Ann(-(n+1)\lambda) \cong \mathbb{C}[\lambda]/ < \lambda^n > \cong H^*(W, \mathbb{C})$,
W : generic smooth degree $(n+1)$ C.-Y. $\mathbb{P}^n \Leftarrow$ polar polyhedron $\Delta(F)^*$. Mirror to $\bar{Y}_s$.

$$\Psi(s,\lambda) = \sum_{m \geq 0} \frac{\Gamma((n+1)(m+\lambda))}{\Gamma(m+\lambda+1)^{n+1}} s^{m+\lambda} =$$

$$\int_{c_0 - i\infty}^{c_0 + i\infty} \frac{\Gamma((n+1)(z+\lambda)+1)}{\Gamma(z+\lambda+1)^{n+1}} (-1)^{z+1} \Gamma(-z)\Gamma(1+z) s^{z+\lambda} \frac{dz}{2\pi i},$$

$$\Psi(s,\lambda) = \psi_0(s) + \psi_1(s)\lambda + \cdots + \psi_n(s)\lambda^n \ in\ \mathcal{R}_{A,\mathcal{T}} \otimes \mathcal{O},$$

$$\Psi(s,\lambda) = \psi_0(s) + \psi_1(s)\lambda + \cdots + \psi_{n-1}(s)\lambda^{n-1} \ in\ \bar{\mathcal{R}}_{A,\mathcal{T}} \otimes \mathcal{O}.$$

The monodromy h_0 of $\Psi(s,\lambda)$ in $\bar{\mathcal{R}}_{A,\mathcal{T}} \otimes \mathcal{O}$ around $s = 0$ is given by

$$\Psi(s,\lambda) \longrightarrow e^{2\pi i\lambda}\Psi(s,\lambda).$$

In other words h_0 acts on $\Psi(s,\lambda)$ by

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2\pi i & 1 & 0 & \cdots & 0 \\ (2\pi i)^2/2 & 2\pi i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (2\pi i)^{n-1}/(n-1)! & * & * & \cdots & 1 \end{pmatrix} \begin{pmatrix} \psi_0(s) \\ \psi_1(s) \\ \vdots \\ \psi_{n-1}(s) \end{pmatrix},$$

Maximally unipotent monodromy around $s = 0$.

Monodromy h_1 around $s=(n+1)^{n+1}$:
$h_1 : \Psi(s,\lambda)$

$$\longrightarrow \Psi(s,\lambda) - 2\pi \mathbf{i} Res_{z=0} \frac{(1-e(-(n+1)z)}{(1-e(-z))^{n+1}} \Psi(s,z).$$

Todd class of W

$$Todd_W = \frac{1-e^{-(n+1)[D]}}{(n+1)[D]} (\frac{[D]}{1-e^{-[D]}})^{n+1} \ \ mod([D]^n) \ in \ H^*(W)$$

with $[D] \in H^2(W)$. The above residue is equivalent to the $[D]^{n-1}$ part of

$$Todd_W \cdot \Psi(s,[D]/2\pi i) \ in \ H^*(W) \otimes \mathcal{O}.$$

(Kontsevich homological mirror type result)

The monodromy h_1 is pseudo-reflection:

$$rank(h_1 - \mathbb{I}_n) = 1$$

in particular it is a reflection $h_1^2 = \mathbb{I}_n$ for
$dim\ Y = n-1$: even. $G =< h_0, h_1 >\subset O(\frac{n+1}{2}, \frac{n-1}{2})$,
$G \cong^? \mathbb{Z}/2 * \mathbb{Z}/(n+1)$.
$dim\ Y = n-1$: odd $G \subset Sp(n, \mathbb{R})$,
$G \cong^? \mathbb{Z} * \mathbb{Z}/(n+1)$.
The monodromy h_∞ at $s = \infty$ is determined by the relation

$$h_1 h_0 h_\infty = \mathbb{I}_n.$$

$$det(t\mathbb{I}_n - h_\infty) = \frac{t^{n+1} - 1}{t - 1}.$$

Example: $d = 2$.

(P.R. Horja 1999) Consider a $(n+1) \times (n+4)$ matrix for smooth C.Y. $W \subset \tilde{\mathbb{P}}(2q_1, \cdots, 2q_n, 1, 1)$ a blow up obtained torically by adding a vector to the defining fan of $\mathbb{P}(2q_1, \cdots, 2q_n, 1, 1)$ (For $n = 2$)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & -2q_1 & -q_1 \\ 0 & 0 & 1 & 0 & -2q_2 & -q_2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -q & q_1 & q_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix},$$

with $q = 1 + \sum_{j=1}^{n} q_j$.

$$\mathcal{R}_{A,\mathcal{T},\mathbb{C}} \cong \mathbb{C}[\boldsymbol{\lambda}]/ < \lambda_1^n(\lambda_1 - 2\lambda_2), \lambda_2^2 > \cong$$
$$H^*(\tilde{\mathbb{P}}(2q_1, \cdots, 2q_n, 1, 1), \mathbb{C}).$$

$$\bar{\mathcal{R}}_{A,\mathcal{T},\mathbb{C}} \cong \mathcal{R}_{A,\mathcal{T},\mathbb{C}}/Ann(-q\lambda_1)$$

$$\cong \mathbb{C}[\lambda]/ < \lambda_1^{n-1}(\lambda_1 - 2\lambda_2), \lambda_2^2 > \cong H^*(W, \mathbb{C}).$$

$\tilde{\Psi}_1(\mathbf{s}, \boldsymbol{\lambda}) \in$ Sol (A-GKZ HGS): periods of Y_s
$\in \bar{\mathcal{R}}_{A,\mathcal{T},\mathbb{C}} \otimes \mathcal{O}_{\mathcal{V}_1}$.
$\tilde{\Psi}_1(\mathbf{s}, \boldsymbol{\lambda}) = \sum_{\mathbf{m} \in (\mathbb{Z}_{\geq 0})^2}$
$$\frac{\Gamma(q(m_1+\lambda_1))s^{\mathbf{m}+\boldsymbol{\lambda}}}{\prod_{j=1}^n \Gamma(q_j(m_1+\lambda_1)+1)\Gamma(m_1+\lambda_1-2(m_2+\lambda_2)+1)\Gamma(m_2+\lambda_2+1)^2}.$$
with $q = 1 + \sum_{j=1}^n q_j$.

Discriminantal divisor Δ_0 :

$$\Delta_0 = \{\mathbf{s}; s_2 = \frac{1}{4}(1 - \frac{\prod_{j=1}^n q_j^{q_j}}{q^q}\frac{1}{s_1})^2\}.$$

Singular loci

$$\Delta_0 \cup \{s_1 = 0\} \cup \{\frac{1}{s_1} = 0\} \cup \{s_2 = 0\} \cup \{s_2 = 1/4\}.$$

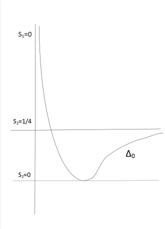
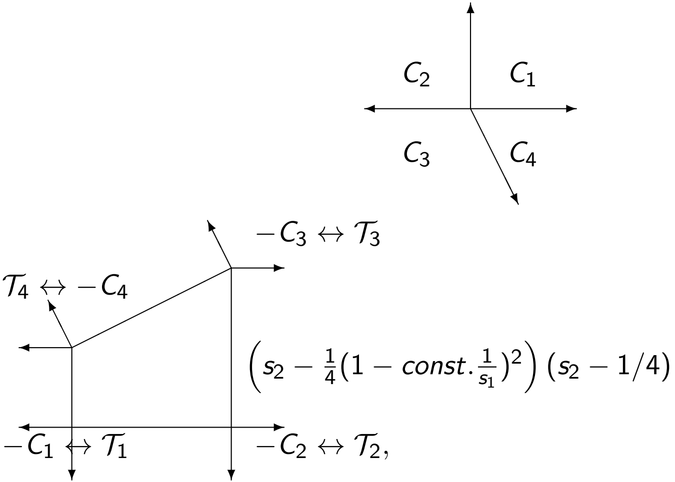
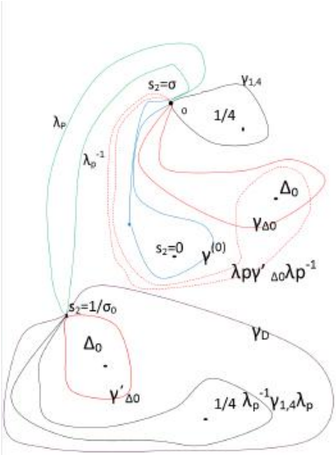


Figure: Singular loci

Secondary polytope, secondary fan



Paths avoiding discriminantal loci



$\Psi_1(\mathbf{s},\boldsymbol{\lambda}) = \tilde{\Psi}_1(e^{-q\pi\mathbf{i}}s_1,\mathbf{s}') = (\frac{1-e(-q\lambda_1)}{2\pi\mathbf{i}})^{-1}\Phi_1(\mathbf{s},\boldsymbol{\lambda})$
Loop action $C_1 \to C_4 \to C_1$.

The first loop (5.2) $\gamma_{1,4}\cdot\gamma^{(0)}\cdot\gamma_{\Delta_0}$.
$\gamma_{1,4} : \Psi_1(\mathbf{s},\boldsymbol{\lambda}) \to \Psi_1(\mathbf{s},\boldsymbol{\lambda}) -$
$$2\pi\mathbf{i}e(\lambda_1)Res^{+}_{\zeta_2}\frac{(1-e(\lambda_1-2\lambda_2)e(-\zeta_2))}{(1-e(-\zeta_2))^2}\Psi_1(\mathbf{s},\lambda_1,\zeta_2)$$

$\gamma^{(0)} : \Psi_1(\mathbf{s},\boldsymbol{\lambda}) \to e(\lambda_2)\Psi_1(\mathbf{s},\boldsymbol{\lambda})$

$$\gamma_{\Delta_0} : \Psi_1(\mathbf{s},\boldsymbol{\lambda}) \to \Psi_1(\mathbf{s},\boldsymbol{\lambda}) - \int_{\Gamma_1}\int_{\Gamma_2} T(\zeta)\Psi_1(\mathbf{s},\boldsymbol{\lambda})d\zeta$$
$T(\zeta)$: unknown kernel.

$$\gamma_{\Delta_0} : \Psi_1(\mathbf{s}, \boldsymbol{\lambda}) \rightarrow \Psi_1(\mathbf{s}, \boldsymbol{\lambda}) - \int_{\Gamma_1}\int_{\Gamma_2} T(\zeta)\Psi_1(\mathbf{s}, \boldsymbol{\lambda})d\zeta$$

$T(\zeta)$: unknown kernel.

$$(\gamma_{1,4} \cdot \gamma^{(0)} \cdot \gamma_{\Delta_0})_* \Psi_1(\mathbf{s}, \boldsymbol{\lambda}) =$$

$$e(\lambda_2)[\Psi_1(\mathbf{s}, \boldsymbol{\lambda}) - 2\pi\mathbf{i}(1-e(\lambda_1-2\lambda_2))Res^+_{\zeta_2}\frac{\Psi_1(\mathbf{s}, \zeta)}{(1-e(-\zeta_2))^2}$$
$$-e(\lambda_1 - 2\lambda_2)\int_{\Gamma_1}\int_{\Gamma_2} T(\zeta)\Psi_1(\mathbf{s}, \boldsymbol{\lambda})d\zeta]. \qquad (5.1)$$

Relations between loops avoiding discriminantal loci

First we remark

$$\lambda_P \cdot \gamma'_{\Delta_0} \cdot (\lambda_P)^{-1} = \gamma^{(0)} \cdot \gamma_{\Delta_0} \cdot (\gamma^{(0)})^{-1}$$

and

$$\gamma_D = \lambda_P^{-1} \cdot \gamma_{1,4} \cdot \lambda_P \cdot \gamma'_{\Delta_0}.$$

The following relations hold:

$$\gamma_{1,4} \cdot \gamma^{(0)} \cdot \gamma_{\Delta_0} \tag{5.2}$$

$$= \gamma_{1,4} \cdot \gamma^{(0)} \cdot \gamma_{\Delta_0} \cdot (\gamma^{(0)})^{-1} \cdot \gamma^{(0)}$$

$$= \gamma_{1,4} \cdot \lambda_P \cdot \gamma'_{\Delta_0} \cdot (\lambda_P)^{-1} \cdot \gamma^{(0)}$$

$$= \lambda_P \cdot \gamma_D \cdot (\lambda_P)^{-1} \cdot \gamma^{(0)}. \tag{5.3}$$

Loop action $C_1 \to C_2 \to C_3 \to C_2 \to C_1 \to C_1$.

The second loop (5.3) $\lambda_p \cdot \gamma_D \cdot (\lambda_p)^{-1} \cdot \gamma^{(0)}$.
$\lambda_p : C_1 \to C_2$

$(\lambda_p)_* \Psi_1(\mathbf{s}, \boldsymbol{\lambda}) =$
$(1 - e(-q\lambda_1)) Res^{-}_{z_1} \dfrac{2\pi \mathbf{i}}{(1 - e(z_1 - \lambda_1))} \Psi_1(\mathbf{s}, z_1, \lambda_2)$

$$\gamma_D = \lambda_p^{-1} \cdot \gamma_{1,4} \cdot \lambda_p \cdot \gamma'_{\Delta_0} : C_2 \to C_3 \to C_2$$

$(\gamma_D)_* \Psi_1(\mathbf{s}, z_1, \lambda_2) =$
$\Psi_1(\mathbf{s}, z_1, \lambda_2) - 2\pi \mathbf{i}(1 - e(z_1 - 2\lambda_2)) Res^{+}_{\zeta_2} \frac{\Psi_1(\mathbf{s}, z_1, \zeta_2)}{(1 - e(-\zeta_2))^2}$

Loop: $\lambda_p^{-1} \cdot \gamma^{(0)} : C_2 \rightarrow C_1 \rightarrow C_1$
$((\lambda_p)^{-1} \cdot \gamma^{(0)})_* \Psi_1(\mathbf{s}, z_1, \zeta_2) =$

$$Res^+_{\zeta_1} \frac{2\pi \mathbf{i}}{(1 - e(\zeta_1 - z_1))} \prod_{j=1}^{n} \left(\frac{(1 - e(-q_j z_1))}{(1 - e(-q_j \zeta_1))} \right)$$
$$\frac{(1 - e(-z_1 + 2\zeta_2))(1 - e(-q\zeta_1))}{(1 - e(-\zeta_1 + 2\zeta_2))} \Psi_1(\mathbf{s}, \zeta).$$

The composition of four paths = the second loop
$(\lambda_p \cdot \gamma_D \cdot (\lambda_p)^{-1} \cdot \gamma^{(0)})_* \Psi_1(\mathbf{s}, \boldsymbol{\lambda}) =$

$$e(\lambda_2)\Psi_1(\mathbf{s}, \boldsymbol{\lambda})$$
$$-2\pi\mathbf{i}e(\lambda_2)(1 - e(\lambda_1 - 2\lambda_2))Res^+_{\zeta_2}\frac{\Psi_1(\mathbf{s}, \zeta)}{(1 - e(-\zeta_2))^2}$$
$$-(2\pi\mathbf{i})^2 e(\lambda_2)e(\lambda_1 - 2\lambda_2)Res^+_{\zeta_2}\frac{1}{(1-e(-\zeta_2))^2}$$

$$\cdot Res^+_{\zeta_1}\frac{(1 - e(-q\zeta_1))}{(1 - e(-(\zeta_1 - 2\zeta_2)))\prod_{j=1}^n(1 - e(-q_j\zeta_1))}\Psi_1(\mathbf{s}, \zeta).$$

Upshot: unknown kernel (5.1)

$$T(\zeta) =$$
$$\frac{(1 - e(-q\zeta_1))}{(1 - e(-\zeta_2))^2\,(1 - e(-(\zeta_1 - 2\zeta_2)))\prod_{j=1}^n(1 - e(-q_j\zeta_1))}.$$

Monodromy around the discriminantal divisor Δ_0 is given by

$$\Psi_1(\mathbf{s},\boldsymbol{\lambda}) \to \Psi_1(\mathbf{s},\boldsymbol{\lambda}) - (2\pi\mathbf{i})^2 Res_{\mathbf{z}=\mathbf{0}} T(\zeta)\Psi_1(\mathbf{s},\zeta)$$

$$T(\zeta) = \frac{1-e(-q\zeta_1)}{(1-e(-\zeta_1+2\zeta_2))(1-e(-\zeta_2))^2\prod_{j=1}^{n}(1-e(-q_j\zeta_1))}$$

$$(2\pi\mathbf{i})^2 Res_{\zeta=\mathbf{0}} T(\zeta)\Psi_1(\mathbf{s},\zeta) = \int_W Todd_W([\mathbf{D}])\Psi_1(s,[\mathbf{D}]/2\pi i).$$

$$Todd_W([\mathbf{D}]) = Todd_W([D_1],[D_2]) =$$

$$= \frac{1-e^{-q[D_1]}}{q[D_1]}\frac{[D_1]-2[D_2]}{1-e^{-[D_1]+2[D_2]}}\left(\frac{[D_2]}{1-e^{-[D_2]}}\right)^2\prod_{j=1}^{n}\frac{q_j[D_1]}{1-e^{-q_j[D_1]}}$$

with $[D_1],[D_2] \in H^2_{toric}(W)$. (Kontsevich homological mirror type result).

Other cases 1.

1. Bidegree (n, m) C.Y. variety in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ with discriminantal divisor.

$$x = \frac{s^n}{(ns+mt)^n}, \quad y = \frac{t^m}{(ns+mt)^m} \tag{5.4}$$

with $[s:t] \in \mathbb{P}^1$. By a covering

$$\begin{array}{cccc} \mathbb{C}^2 & \longrightarrow & & \mathbb{C}^2 \\ (\xi, \eta) & \longrightarrow & (x, y) = & (\xi^n, \eta^m) \end{array} \tag{5.5}$$

it becomes a complex hyperplane arrangement (generally true for every multi-degree C.Y. variety)

$$\{(\xi, \eta) \in \mathbb{C}^2; \xi\eta \prod_{0 \leq i \leq n-1, 0 \leq j \leq m-1} (n\omega(n)^i \xi + m\omega(m)^j \eta - 1) = 0\}.$$

with $\omega(n)^n = \omega(m)^m = 1$.

Other cases 2.

2. The self mirror variety

$$\sum_{j=1}^{n}(x_j+\frac{t_j}{x_j})+1=0.$$

with the discriminantal divisor (same as that for Lauricella F_C)

$$t_j=(\frac{s_j}{\sum_{\ell=1}^{n} s_\ell})^2, \quad 1\leq j\leq n \tag{5.6}$$

with $[s_1:\cdots,:s_n]\in\mathbb{P}^{n-1}$. The 2^n covering $u_j^2=t_j, 1\leq j\leq n$ becomes a real hyper plane arrangement.