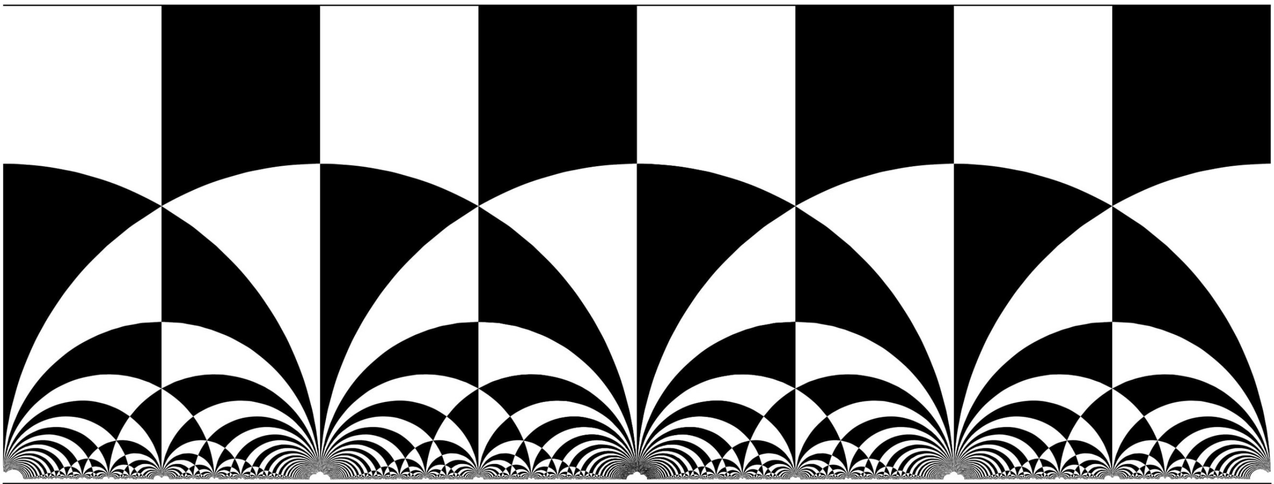

GALATASARAY UNIVERSITY — MATHEMATICS DEPARTMENT

SECOND

ROMANIAN-TURKISH MATHEMATICS COLLOQUIUM

25-29 OCTOBER 2017
ISTANBUL, TURKEY

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AND

ROMANIAN–TURKISH JOINT LABORATORY OF MATHEMATICAL RESEARCH

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Preface

First Romanian-Turkish Mathematics Colloquium held at the Faculty of Mathematics and Informatics, Ovidius University of Constanta, Romania in 2015 . The Second Romanian-Turkish Mathematics Colloquium was organised in Galatasaray University, Istanbul, in collaboration with Institute of Mathematics of the Romanian Academy and Romanian–Turkish Joint Laboratory of Mathematical Research. The aim is to foster collaboration between the mathematicians who study on various fields of mathematics, from the two countries .

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10:00-10:10	Break			
10:10-10:40	Lucian Beznea IMAR		Mustafa Gökhan Benli METU	
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14:00-14:30	Paul Voutier London	Derya Çıray U. Constanz	Hatice Boylan İstanbul U.	Ayberk Zeytin Galatasaray U.
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Operator Models for Hilbert Locally C^*- Modules

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Abstract

We single out the concept of represented (concrete) Hilbert locally C^*-module by locally bounded operators, then show in Theorem that this concept makes the operator model for all Hilbert locally C^*-modules and, as an application, we obtain in Theorem a direct construction of the exterior tensor product of Hilbert locally C^*-modules. These are obtained as consequences of a general dilation theorem for positive semidefinite kernels with values locally bounded operators, presented in both linearisation (Kolmogorov decomposition) form and reproducing kernel space form. More precisely, we obtain in Theorem a rather general dilation theorem for positive semidefinite kernels with values locally bounded operators and that are invariant under a left action of a $*$-semigroup.

Preliminaries

By definition, $\{{}_\lambda\}_{\lambda\in\Lambda}$ is a *strictly inductive system of Hilbert spaces* if

(lhs1) $(\Lambda;\leq)$ is a directed poset;

(lhs2) $\{{}_\lambda\}_{\lambda\in\Lambda}$ is a net of Hilbert spaces $({}_\lambda;\langle\cdot,\cdot\rangle_\lambda)$, $\lambda\in\Lambda$;

(lhs3) for each $\lambda,\mu\in\Lambda$ with $\lambda\leq\mu$ we have ${}_\lambda\subseteq{}_\mu$;

(lhs4) for each $\lambda,\mu\in\Lambda$ with $\lambda\leq\mu$ the inclusion map $J_{\mu,\lambda}:\ {}_\lambda\rangle_\mu$ is isometric, that is,

$$\langle x,y\rangle_\lambda=\langle x,y\rangle_\mu,\ \text{for all } x,y\in{}_\lambda\ . \tag{1}$$

For any strictly inductive system of Hilbert spaces $\{{}_\lambda\}_{\lambda\in\Lambda}$, its inductive limit $=\varinjlim_{\lambda\in\Lambda}{}_\lambda$ is a Hausdorff locally convex space.

A *locally Hilbert space*, see [4], [5], [3], is, by definition, the inductive limit

$$=\varinjlim_{\lambda\in\Lambda}{}_\lambda=\bigcup_{\lambda\in\Lambda}{}_\lambda, \tag{2}$$

of a strictly inductive system $\{{}_\lambda\}_{\lambda\in\Lambda}$ of Hilbert spaces. We stress the fact that, a locally Hilbert space is rather a special type of locally convex space and, in general, not a Hilbert space. It is clear that a locally Hilbert space is uniquely determined by the strictly inductive system of Hilbert spaces.

Let $=\varinjlim_{\lambda\in A}{}_\lambda$ and ${}^r=\varinjlim^r_{\lambda\in A}{}_\lambda$ be two locally Hilbert spaces generated by strictly inductive systems of Hilbert spaces $(\{{}_\lambda\}_{\lambda\in\Lambda};\{J_{\mu,\lambda}\}_{\lambda\leq\mu})$ and, respectively, $(\{{}^r_\lambda\}_{\lambda\in\Lambda};\{J^r_{\mu,\lambda}\}_{\lambda\leq\mu})$, indexed on the same directed poset Λ. A linear map $T:\ \rangle^r$ is called a *locally bounded operator* if T is a continuous coherent linear map and adjointable, more precisely,

(lbo1) There exists a net of operators $\{T_\lambda\}_{\lambda\in\Lambda}$, with $T_\lambda\in({}_\lambda,{}^r_\lambda)$ such that $TJ_\lambda=J^r_\lambda T_\lambda$ for all $\lambda\in\Lambda$.

(lbo2) The net of operators $\{T^*_\lambda\}_{\lambda\in\Lambda}$ is coherent as well, that is, $T^*_\mu J^r_{\mu,\lambda}=J_{\mu,\lambda}T^*_\lambda$, for all $\lambda,\mu\in\Lambda$ such that $\lambda\leq\mu$.

We denote by $(,^r)$ the collection of all locally bounded operators $T:\ \rangle^r$. It is easy to see that $(,^r)$ is a vector space.

Positive Semidefinite Kernels

Let X be a nonempty set and $= \varinjlim_{\lambda\in\Lambda} {}_{\lambda}$ be a locally Hilbert space, for some directed poset Λ. A map $: X \times X\rangle()$ is called a *locally bounded operator valued kernel* on X. Equivalently, there exists a projective system $\{{}_{\lambda} \mid \lambda \in \Lambda\}$ of kernels ${}_{\lambda} : X \times X\rangle({}_{\lambda})$, $\lambda \in \Lambda$, where

$$ {}_{\lambda}(x,y) = (x,y)_{\lambda}, \quad \lambda \in \Lambda,\ x,y \in X, \tag{3} $$

more precisely, for each $\lambda \in \Lambda$ we have ${}_{\lambda}(x,y) \in ({}_{\lambda})$ such that

$$ {}_{\lambda}(x,y)P_{\lambda,\mu} = P_{\lambda,\mu\,\lambda}(x,y), \quad x,y \in X,\ \lambda \leq \mu, \tag{4} $$

where $P_{\lambda,\mu}$ is the orthogonal projection of ${}_{\mu}$ onto ${}_{\lambda}$, and, for any $h \in$,

$$ (x,y)h =_{\lambda} (x,y)h, \quad x,y \in X, \tag{5} $$

where $\lambda \in \Lambda$ is such that $h \in_{\lambda}$.

Given $n \in$, the kernel $: X \times X\rangle()$ is called *n-positive semidefinite* if, for any $x_1,\ldots,x_n \in X$ and any $h_1,\ldots,h_n \in$, we have

$$ \sum_{i,j=1}^{n} \langle (x_i,x_j)h_j, h_i\rangle_{\geq} 0. \tag{6} $$

It is easy to see that is n-positive semidefinite if and only if, for each $\lambda \in \Lambda$, the kernel ${}_{\lambda}$ is n-positive semidefinite.

The kernel $: X \times X\rangle()$ is called *positive semidefinite* if it is n-positive semidefinite for all $n \in$. Clearly, this is equivalent with the condition that, for each $\lambda \in \Lambda$, the kernel ${}_{\lambda}$ is positive semidefinite.

Given a locally bounded operator valued kernel $: X \times X\rangle()$, with $= \varinjlim_{\lambda\in\Lambda} {}_{\lambda}$, a *locally Hilbert space linearisation*, also called a *locally Hilbert space Kolmogorov decomposition*, of is a pair $({}^{r};V)$ such that

(l1) ${}^{r} = \varinjlim_{\lambda\in\Lambda}^{r} {}_{\lambda}$ is a locally Hilbert space over the same directed poset Λ.

(l2) $V : X\rangle(,{}^{r})$ has the property $(x,y) = V(x)^{*}V(y)$, for all $x,y \in X$.

A linearisation $({}^{r};V)$ of is called *minimal* if

(l3) $V(X)$ is a total subset in ^{r}.

Let S be a $*$-semigroup acting on X at left, $S \times X \ni (s,x) \mapsto s \cdot x \in X$. A kernel $: X \times X\rangle()$, for some locally Hilbert space , is called *S-invariant* if

$$ (s \cdot x, y) = (x, s^{*} \cdot y), \quad s \in S,\ x,y \in X. \tag{7} $$

Invariant kernels and their many applications have been considered in mathematical models of quantum physics [2] and (quantum) probability theory [7].

The following theorem characterises those invariant positive semidefinite kernels that yield $*$-representations of the $*$-semigroup on the dilation space.

Theorem. *Let S be a $*$-semigroup acting at left on the nonempty set X and let $: X \times X\rangle()$ be a kernel, for some locally Hilbert space $= \varinjlim_{\lambda\in\Lambda} {}_{\lambda}$. The following assertions are equivalent:*

(1) *The kernel is locally positive semidefinite, invariant under the action of S, and*

  (b) *For any $s \in S$ and any $\lambda \in \Lambda$, there exists $c_{\lambda}(s) \geq 0$ such that, for any $n \in$, any vectors $h_1,\ldots,h_n \in_{\lambda}$, and any elements $x_1,\ldots,x_n \in X$, we have*

$$ \sum_{j,k=1}^{n} \langle (s\cdot x_j, s\cdot x_k)h_k, h_j\rangle_{\lambda} \leq c_{\lambda}(s) \sum_{j,k=1}^{n} \langle (x_j,x_k)h_k, h_j\rangle_{\lambda}. $$

(2) *There exists a triple* $({}^r;\pi;V)$ *subject to the following properties:*

 (il1) $({}^r;V)$ *is a locally Hilbert space linearisation of .*
 (il2) $\pi\colon S\rangle({}^r)$ *is a* $*$*-representation.*
 (il3) $V(s\cdot x)=\pi(s)V(x)$ *for all* $s\in S$ *and all* $x\in X$.

(3) *There exists a reproducing kernel locally Hilbert space* $\mathscr{R}$ *with reproducing kernel and a* $*$*-representation* $\rho\colon S\rangle(\mathscr{R})$ *such that* ${}_{s\cdot x}=\rho(s)_x$ *for all* $s\in S$ *and all* $x\in X$.

In addition, if this is the case, then the triple $({}^r;\pi;V)$ *as in item* (2) *can be chosen minimal, in the sense that* $\pi(S)V(X)$ *is total in* ^{r} *and, in this case, it is unique up to a locally unitary equivalence.*

Hilbert Locally C^*-Modules

The origins of Hilbert modules over locally C^*-algebras (shortly, Hilbert locally C^*-modules) are related to investigations on noncommutative analogues of classical topological objects (groups, Lie groups, vector bundles, index of elliptic operators, etc.) as seen in W.B. Arveson [1], A. Mallios [6], D.V. Voiculescu [9], N.C. Phillips [8], to name a few.

We first briefly review the abstract concepts related to Hilbert modules over locally C^*-algebras, see [6], [8], [10]. Let be a locally C^*-algebra and let be a complex vector space. A paring $[\cdot,\cdot]\colon \times\rangle$ is called an *-valued gramian* or *-valued inner product* if

(g1) $[e,e]\geq 0$ for all $e\in$, and $[e,e]=0$ if and only if $e=0$.
(g2) $[e,\alpha g+\beta f]=\alpha[e,g]+\beta[e,f]$, for all $\alpha,\beta\in$ and $e,f,g\in$.
(g3) $[e,f]^*=[f,e]$ for all $e,f\in$.

The vector space is called a *pre-Hilbert locally* C^**-module* if

(h1) On there exists an -gramian $[\cdot,\cdot]$, for some locally C^*-algebra .
(h2) is a right -module compatible with the -vector space structure of .
(h3) $[e,af]=[e,f]a$ for all $a\in$ and all $e,f\in$.

On any pre-Hilbert locally C^*-module over the locally C^*-algebra , with -gramian $[\cdot,\cdot]$, there exists a natural Hausdorff locally convex topology. More precisely, for any $p\in S()$, that is, p is a continuous C^*-seminorm on , letting

$$p(e)=p([e,e])^{1/2},\quad e\in, \tag{8}$$

then p is a seminorm on . If the topology generated on by $\{p\mid p\in S()\}$ is complete, then is called a *Hilbert locally* C^**-module*. In case is a C^*-algebra, we talk about a *Hilbert* C^**-module* , with norm $\ni e\mapsto \|[e,e]\|_{1/2}$.

Let be a Hilbert module over a locally C^*-algebra and, for $p\in S()$, recall that ${}_p=\ker(p)$ is a closed $*$-ideal of with respect to which ${}_p=/{}_p$ becomes a C^*-algebra under the canonical quotient C^*-norm $\|\cdot\|_p$. Considering

$$\mathscr{N}_p=\{e\in\mid [e,e]\in{}_p\}, \tag{9}$$

then $\mathscr{N}_p$ is a closed -submodule of and ${}_p=/\mathscr{N}_p$ is a Hilbert module over the C^*-algebra ${}_p$, with norm

$$\|e+\mathscr{N}_p\|_p=\inf_{f\in\mathscr{N}_p}p(e+f)=p(e),\quad e\in . \tag{10}$$

For each $p,q\in S()$ with $p\leq q$, observe that $\mathscr{N}_q\subseteq\mathscr{N}_p$ and hence, there exists a canonical projection $\pi_{p,q}\colon {}_q\rangle{}_p$, $\pi_{p,q}(e+\mathscr{N}_q)=e+\mathscr{N}_p$, $h\in$, and $\pi_{p,q}$ is an -module map, such that $\|\pi_{p,q}(e+\mathscr{N}_q)\|_p\leq\|e+\mathscr{N}_p\|_q$ for all $e\in$. In addition, $\{{}_p\}_{p\in S()}$ and $\{\pi_{p,q}\mid p,q\in S(),\ p\leq q\}$ make a projective system of Hilbert C^*-modules and $=\varprojlim_{p\in S()}{}_p$.

(1) Let $= \varinjlim_{\lambda\in\Lambda} {}_\lambda$ *and* ${}^r = \varinjlim^{r}_{\lambda\in\Lambda} {}_\lambda$ *be two locally Hilbert spaces with respect to the same directed poset* Λ*. We consider* $()$ *as a locally* C^**-algebra. Observe that the vector space* $({,}^r)$ *has a natural structure of right* $()$*-module which is compatible with the -vector space structure of* $({,}^r)$ *and, considering the gramian* $[\cdot,\cdot]_{({,}^r)}$ *defined by*

$$[T,S]_{({,}^r)} = T^*S, \quad T,S \in ({,}^r), \tag{11}$$

$({,}^r)$ *becomes a pre-Hilbert module over the locally* C^**-algebra* $()$*.*

The complex vector space $({,}^r)$ *has a natural family of seminorms*

$$q_\mu(T) = \|T_\mu\|_{({}_\mu,{}^r_\mu)}, \quad T = \varprojlim_{\lambda\in\Lambda} T_\lambda \in ({,}^r),\ \mu\in\Lambda. \tag{12}$$

Observe that, with respect to the C^**-seminorms* p_μ *on* $()$ *we have*

$$q_\mu(T)^2 = \|T_\mu\|^2_{({}_\mu,{}^r_\mu)} = \|T_\mu^* T_\mu\|_{({}_\mu)} = p_\mu([T,T]_{({,}^r)}),\ \mu\in\Lambda,\ T = \varprojlim_{\lambda\in\Lambda} T_\lambda \in ({,}^r),$$

hence, compare with (8)*, the collection of seminorms* $\{q_\mu\}_{\mu\in\Lambda}$ *defines exactly the canonical topology on the pre-Hilbert locally* C^**-module* $({,}^r)$*. Since, as easily observed, this locally convex topology is complete on* $({,}^r)$*, it follows that* $({,}^r)$ *is a Hilbert locally* C^**-module over* $()$*.*

(2) With notation as in item (1), let be a closed $*$*-subalgebra of* $()$*, considered as a locally* C^**-algebra. Let be a closed vector subspace of* $({,}^r)$ *that is an -module and such that* $T^*S \in$ *for all* $T,S\in$*. Then, the definition in* (11) *provides a gramian* $[T,S]_{=}T^*S$*,* $T,S\in$*, which turns into a Hilbert locally* C^**-module over . Observe that the embedding of in* $({,}^r)$ *is a coherent linear map.*

A Hilbert locally C^*-module as in Example .(2) is called a *represented Hilbert locally* C^**-module* or, a *concrete Hilbert locally* C^**-module*. As an application of the dilation theorem for invariant positive semidefinite kernels, we have the following theorem that says that represented Hilbert locally C^*-modules make the universal model for all Hilbert locally C^*-modules.

Theorem. *Let be a Hilbert module over some locally* C^**-algebra . Then, is isomorphic to a concrete Hilbert locally* C^**-module, more precisely, there exist two locally Hilbert spaces* $= \varinjlim_{p\in S()} {}_p$ *and* ${}^r = \varinjlim^{r}_{p\in S()} {}_p$*, a coherent faithful* $*$*-morphism* $\phi\colon \rangle()$*, and a coherent one-to-one linear map* $\Phi\colon \rangle({,}^r)$ *such that:*

(i) $\Phi(e)^*\Phi(f) = \phi([e,f]_{})$ *for all* $e,f\in$*.*
(ii) $\Phi(ea) = \Phi(e)\phi(a)$ *for all* $e\in$ *and all* $a\in$*.*

Let and be two locally C^*-algebras and let and be two Hilbert locally C^*-modules over and, respectively, . Let $\otimes_*$ denote the spatial locally C^*-algebra tensor product. Consider the algebraic tensor product $\otimes_{\mathrm{alg}}$ of the vector spaces and and observe that there is a natural right action of the (algebraic) tensor product $*$-algebra $\otimes_{\mathrm{alg}}$, first defined on elementary tensors

$$(e\otimes f)(a\otimes b) = (ea)\otimes(fb), \quad a\in,\ b\in,\ e\in,\ f\in, \tag{13}$$

and then extended by linearity, hence $\otimes_{\mathrm{alg}}$ is naturally an $\otimes_{\mathrm{alg}}$-module. Also, there is an $\otimes_{\mathrm{alg}}$-valued pairing on $\otimes_{\mathrm{alg}}$, firstly defined on elementary tensors

$$[e_1\otimes f_1, e_2\otimes f_2] = [e_1,f_1]\otimes[e_2,f_2], \quad e_1,e_2\in,\ f_1,f_2\in, \tag{14}$$

and then extended by linearity.

Theorem. *With notation as before, the pairing defined at* (14) *is uniquely extended to an* $\otimes_*$*-gramian on* $\otimes_{\mathrm{alg}}$*, with respect to which it is a pre-Hilbert locally* C^**-module, and then it is uniquely extended to the completion of* $\otimes_{\mathrm{alg}}$ *to a Hilbert module over the locally* C^**-algebra* $\otimes_*$*.*

Acknowledgments

Talk based on the article A. GHEONDEA, Operator Models for Hilbert Locally C^*-Modules, *Operators and Matrices*, **11**(2017), 639–667.

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A higher-rank variation of Artin's primitive root conjecture

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Abstract

Let $\Gamma \subset \mathbb{Q}^*$ be a finitely generated subgroup and let p be a prime number such that the reduction group Γ_p is a well defined subgroup of the multiplicative group $^*_p$, for any $m \in \mathbb{N}$ we present an asymptotic formula for the average of the number of primes $p \leq x$ for which the index $[^*_p\colon \Gamma_p] = m$. The average is performed over all finitely generated subgroups $\Gamma = \langle a_1, \ldots, a_r \rangle \subset \mathbb{Q}^*$, with $a_i \in \mathbb{Z}$ and $a_i \leq T_i$ with a range of uniformity: $T_i > \exp(4(\log x \log\log x)^{\frac{1}{2}})$ for every $i = 1, \ldots, r$. The case of rank 1 and $m = 1$ corresponds to the classical Artin conjecture for primitive roots and has already been considered by P. J. Stephens in 1969. This presentation is a short version of a joint work with Lorenzo Menici [7].

Terminology

Throughout the text, the letter p always denote *prime numbers*. As usual, we use $\pi(x)$ to denote the number of primes up to x and

$$(x) = \int_2^x \frac{dt}{\log t}$$

denotes the *logarithmic integral* function.

φ and μ are respectively the *Euler* and the *Möbius* functions. $\omega(n)$ denotes the number of distinct prime factors of n and (n) denotes the product of distinct prime numbers dividing n. For functions F and $G > 0$ the notations $F = O(G)$ and $F \ll G$ are equivalent to the assertion that the inequality $|F| \leq c\,G$ holds with some constant $c > 0$. In what follows, all constants implied by the symbols O and $\ll$ may depend parameter, we write O_λ and $\ll_\lambda$ to indicate that the implied constant depends on a given parameter λ.

Let $\Gamma \subseteq \mathbb{Q}^*$ be a finitely generated multiplicative subgroup. The *support* of Γ is the (finite) set of prime numbers p for which the p–adic valuation $v_p(g) \neq 0$ for some $g \in \Gamma$. We denote this set by $\operatorname{supp}\Gamma$ and define $\sigma_\Gamma = \prod_{p \in \operatorname{supp}\Gamma} p$. For each prime number $p \nmid \sigma_\Gamma$, the reduction of Γ modulo p is well defined. That is,

$$\Gamma_p = \{g \pmod p : g \in \Gamma\}.$$

Introduction

In his classic work *Disquisitones Arithmeticae,* Carl F. Gauss questioned why the rational number $\frac{1}{7}$ has a period of length 6, whereas $\frac{1}{11}$ has a period of length 2. In the same work, he observed that for any prime number $p \neq 2, 5$, $\frac{1}{p}$ has the same period with the order of 10 mod p, and that the period of $\frac{1}{p}$ is long when 10 is a primitive root modulo p, where an integer a is said to be a primitive root modulo p if $\langle ap \rangle =^*_p$. With this observation, Gauss further questioned whether 10 would be a primitive root for infinitely many prime numbers.

Later on, in 1927, Artin [1] conjectured that any non-zero integer $a \neq \pm 1$, which is not a perfect square, is a primitive root for infinitely many primes. Letting p be a prime number, and denoting the multiplicative order of an integer a modulo p by $\ell_a(p)$, we say that the integer a is primitive root modulo p if $\ell_a(p) = p-1$. Also defining $N_a(x)$ as the number of primes up to x for which $\ell_a(p) = p-1$, we may formulate Artin's initial conjecture as: Let a be a fixed integer such that $a \neq \pm 1, 0$ or a perfect square. Write $a = b^h$ where $b \in \mathbb{Z}$ is not a perfect power and $h \in \mathbb{N}$. Then,

$$N_a(x) \sim A_h \frac{x}{\log x}$$

as $x \to \infty$ where

$$A_h = \prod_{\substack{q|h \\ q \text{ prime}}} \left(1 - \frac{1}{q-1}\right) \prod_{\substack{q\nmid h \\ q \text{ prime}}} \left(1 - \frac{1}{q(q-1)}\right).$$

In 1967, Hooley [9] proved that Artin's conjecture is true and that we may obtain an asymptotic formula for $N_a(x)$, under the additional assumption that GRH (Generalized Riemann Hypothesis) holds.

Theorem. *[9] Suppose $a \in \mathbb{Z} \setminus \{\pm 1, 0\}$ which is not a perfect square. If the GRH holds for the Dedekind zeta functions for the fields $\mathbb{Q}(\zeta_k, a^{1/k})$ with $k \in \mathbb{N}$ square-free, and where ζ_k is a primitive k-th root of unity, then*

$$N_a(x) = A(a)\pi(x) + O\left(\frac{x \log\log x}{(\log x)^2}\right)$$

where $A(a)$ is a constant depending on a.

Until so far, the best unconditional result about Artin's conjecture is due to Heath-Brown [5], Gupta and Murty [3]: One of 2, 3, or 5 is a primitive root modulo p for infinitely many primes p.

Average works

Let us next move average results on Artin's conjecture without the GRH assumption. Just one year after the work of Hooley, in 1968, Goldfeld [2] showed unconditionally that for each $D > 1$

$$N_a(x) = A \operatorname{li} x + O\left(\frac{x}{(\log x)^D}\right) \tag{1}$$

holds for all integers $a \leq N$ with at most $c_1 N^{\frac{9}{10}} (5 \log x + 1)^{h+D+2}$ exceptions where $h = \frac{x}{\log N}$, A is Artin's constant, c_1 and constant of O-term are positive and depend on only D.

Later in 1969, Stephens [9] not only showed that in average the asymptotic formula 1 holds, but also making use of the normal order method of Turan, he proved that the number of exceptions is bounded by $O(N)$ when

$$N > \exp(6(\log x \log\log x)^{\frac{1}{2}})$$

and as N, x tends to infinity. The following theorems are again due to Stephens and they were used to prove his results just mentioned.

Theorem. *[9] If*

$$N > \exp(4(\log x \log\log x)^{\frac{1}{2}}),$$

then

$$\frac{1}{N} \sum_{a \leq N} N_a(x) = A \operatorname{li} x + O\left(\frac{x}{(\log x)^D}\right),$$

where A is Artin's constant, and the constant $D > 1$ is arbitrary.

Theorem. *[9] Let A be Artin's constant, and $E > 2$ be an arbitrary real number. Then, for*

$$N > \exp(6(\log x \log\log x)^{\frac{1}{2}}),$$

we have

$$\frac{1}{N}\sum_{a\le N}(N_a(x) - A\,\mathrm{li}\,x)^2 \ll \frac{x^2}{(\log x)^E}.$$

For any integer $|a| > 1$ which is not a perfect square, Artin's conjecture is about the number of primes which satisfy the relation $[F_p^* : \langle ap\rangle] = 1$. We can define a new counting function to enumerate the prime numbers which a generates a group of index $m \in \mathbb{N}$ in F_p^*,

$$N_a(x,m) = \#\{p \le x : p \nmid a, [F_p^* : \langle a \mod p\rangle] = m\}. \tag{2}$$

The following theorem was proven by Moree.

Theorem. *[8] Let m be an arbitrary positive integer. Then for $T > \exp(4(\log x \log\log x)^{1/2})$, we have*

$$\frac{1}{T}\sum_{a\le T} N_{a,m}(x) = \sum_{\substack{p\le x\\ p\equiv 1 m}} \frac{\varphi((p-1)/m)}{p-1} + O\left(\frac{x}{(\log x)^E}\right) \tag{3}$$

for any constant $E > 2$.

In a joint work with Lorenzo Menici [7], we generalize the result of Moree given in Theorem and the results of Stephens given in Theorems and to the r–rank case. Also, we prove an asymptotic formula for the average of the number of primes $p \le x$ for which the index $[^*_p : \Gamma_p] = m$. The average is performed over all finitely generated subgroups $\Gamma = \langle a_1,\ldots,a_r\rangle \subset \mathbb{Q}^*$, with $a_i \in \mathbb{Z}$ and $a_i \le T_i$ with a range of uniformity: $T_i > \exp(4(\log x \log\log x)^{\frac{1}{2}})$ for every $i = 1,\ldots,r$. The main result of paper [7] is summarized in the following theorem.

Theorem. *Assume $T^* := \min\{T_i : i = 1,\ldots,r\} > \exp(4(\log x \log\log x)^{\frac{1}{2}})$ and $m \le (\log x)^D$ for an arbitrary positive constant D. Then*

$$\frac{1}{T_1\cdots T_r}\sum_{\substack{a_i\in\mathbb{Z}\\ 0<a_1\le T_1\\ \vdots\\ 0<a_r\le T_r}} N_{\langle a_1,\cdots,a_r\rangle,m}(x) = C_{r,m}\,\mathrm{Li}(x) + O\left(\frac{x}{(\log x)^M}\right),$$

where $C_{r,m} = \sum_{n\ge 1}\frac{\mu(n)}{(nm)^r\varphi(nm)}$ and $M > 1$ is arbitrarily large.
Furthermore, if $T^* > \exp(6(\log x \log\log x)^{\frac{1}{2}})$, then

$$\frac{1}{T_1\cdots T_r}\sum_{\substack{a_i\in\mathbb{Z}\\ 0<a_1\le T_1\\ \vdots\\ 0<a_r\le T_r}} \left\{N_{\langle a_1,\cdots,a_r\rangle,m}(x) - C_{r,m}\,\mathrm{Li}(x)\right\}^2 \ll \frac{x^2}{(\log x)^{M'}},$$

where $M' > 2$ is arbitrarily large.

Moreover, in [7], we define multiple ramanujan sum and prove the following Lemma. Let $q > 1$ be an integer and let $\in \mathbb{Z}^r$. We define the *multiple Ramanujan sum* as

$$c_q() := \sum_{\substack{\in(\mathbb{Z}/q\mathbb{Z})^r\\ (q,)=1}} e^{2\pi i\cdot/q}.$$

It is well known (see [4, Theorem 272]) that, given any integer n,

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)}. \tag{4}$$

In the following Lemma, we prove a similar equation for multiple Ramanujan sum.

[. *Let*

$$J_r(m) := m^r \prod_{\ell | m}\left(1 - \frac{1}{\ell^r}\right)$$

be the Jordan's totient function, then

$$c_q() = \mu\left(\frac{q}{(q,)}\right)\frac{J_r(q)}{J_r\left(\frac{q}{(q,)}\right)}.$$

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Convergence rates for Kaczmarz-type algorithms

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Abstract

We present in our paper some very recent results on convergence rates for Kaczmarz and Extended Kaczmarz type algorithms. We know that for consistent problems, the Kaczmarz-Tanabe projection method (in which a complete set of projections following the rows of the system matrix is performed), as well as Kaczmarz with Remotest set (Maximal Residual) or Random control have a linear convergence rate. For completing these results we show that Kaczmarz alorithm with Almost cyclic control has a sublinear convergence rate. We then extend these results to inconsistent problems and show similar ones for Extended Kaczmarz type algorithms.

Introduction

According to [1], we introduce the following convergence behaviors for sequences of real, n-dimensional vectors.

Definition. *Let $(x^k)_{k\geq 0} \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and $(\varepsilon_k)_{k\geq 0}$ a sequence of positive real numbers such that*

$$\| x^k - \xi \| \leq \varepsilon^k \longrightarrow 0, \text{ for } k \to \infty. \tag{1}$$

One say that the sequence $(x^k)_{k\geq 0}$ converges to ξ with **order** $p \geq 1$ *if (1) holds with*

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^p} = \mu > 0. \tag{2}$$

If $p = 1$, and in addition $\mu < 1$ in (2), we say that $(x^k)_{k\geq 0}$ converges **linearly** *to ξ; if $p = \mu = 1$ in (2) the convergence is called* **sublinear**.

[. *With respect to the above definitions, the sublinear behavior appears when we have the limit equal to 1 in (2) for $p = 1$. However, it may happens (as it will be the case through the present paper) that this does not exactly hold, but the following situation occurs: let $\Delta_k = \frac{\varepsilon_{k+1}}{\varepsilon_k}, \forall k \geq 0$; it exists a subsequence $(\Delta_{k_s})_{s\geq 0}$ of $(\Delta_k)_{k\geq 0}$ such that*

$$\Delta_k = 1, \forall k \neq k_s \text{ and } \Delta_{k_s} = \delta \in [0,1), \forall s \geq 0. \tag{3}$$

We will consider also this case as a sublinear convergence rate.

Convergence rates

We start this section with some definitions and notations which will be used in the rest of the paper. A is an $m \times n$ matrix with $A_i \neq 0$, $A^j \neq 0$ its i-th row, resectively j-th column, and $\hat{b} \in I\!R^m$ a given vector. We consider the linear least squares problem: find $x \in I\!R^n$ such that

$$\| Ax - \hat{b} \| = \min_{z \in I\!R^n} \| Az - \hat{b} \|, \tag{4}$$

and denote by $LSS(A;\hat{b})$, x_{LS} its set of solutions and the minimal norm one. If P_C will denote the orthogonal projection operator onto a convex closed set, and $\mathcal{N}(A)$, $\mathcal{R}(A)$ are the null space and range of the matrix A we know that the elements x of $LSS(A;b)$ are of the form (see e.g. [2])

$$x = P_{\mathcal{N}(A)}(x) + x_{LS}, \tag{5}$$

and x_{LS} is the unique solution which is orthogonal on $\mathcal{N}(A)$. In general

$$\hat{b} = b + r, \;\; b = P_{\mathcal{R}(A)}(\hat{b}), \;\; r = P_{\mathcal{N}(A^T)}(\hat{b}), \tag{6}$$

and if $r = 0$ the problem (4) is nothing else than the (classical consistent) system

$$Ax = b, \tag{7}$$

and $LSS(A;\hat{b})$ will be denoted as $S(A;b)$.
Algorithm Kaczmarz (K)
Initialization: $x^0 \in I\!R^n$
Iterative step: for $k = 0, 1, \ldots$ select $i_k \in \{1, 2, \ldots, m\}$ and compute x^{k+1} as

$$x^{k+1} = x^k - \frac{\langle x^k, A_{i_k} \rangle - b_{i_k}}{\| A_{i_k} \|^2} A_{i_k}. \tag{8}$$

The **almost cyclic** selection procedure for the index i_k is: select $i_k \in \{1, 2, \ldots, m\}$, such that it exists an integer Γ with

$$\{1, 2, \ldots, m\} \subset \{i_{k+1}, \ldots, i_{k+\Gamma}\} \tag{9}$$

for every $k \geq 0$. Its particular **cyclic** case is: set $i_k = k \bmod m + 1$. The algorithm **K** with almost cyclic selection will be called **ACK**.

Result 1. *The ACK algorithm has a sublinear convergence rate.*

Algorithm Extended Kaczmarz (EK)
Initialization: $x^0 \in I\!R^n, y^0 = \hat{b}$
Iterative step: Select the index $j_k \in \{1, \ldots, n\}$ and set

$$y^k = y^{k-1} - \langle y^{k-1}, A^{j_k} \rangle A^{j_k}. \tag{10}$$

Update the right hand side as

$$b^k = \hat{b} - y^k. \tag{11}$$

Select the index $i_k \in \{1, 2, \ldots, m\}$ and compute x^{k+1} as

$$x^k = x^{k-1} - \frac{\langle x^{k-1}, A_{i_k} \rangle - b^k_{i_k}}{\| A_{i_k} \|^2} A_{i_k}. \tag{12}$$

According to the selection procedures used in the above algorithm we distinguish the following cases.

- **Maximal Residual Extended Kaczmarz (MREK)** Select $j_k \in \{1, 2, \ldots, n\}$ and $i_k \in \{1, 2, \ldots, m\}$ such that

$$|\langle A^{j_k}, y^{k-1} \rangle| = \max_{1 \leq j \leq n} |\langle A^j, y^{k-1} \rangle|, \tag{13}$$

$$|\langle A_{i_k}, x^{k-1} \rangle - b^k_{i_k}| = \max_{1 \leq i \leq m} |\langle A_i, x^{k-1} \rangle - b^k_i|. \tag{14}$$

- **Almost cyclic Extended Kaczmarz (ACEK)** Select $j_k \in \{1,2,\ldots,n\}$, $i_k \in \{1,2,\ldots,m\}$, such that there exist integers Γ, Δ with

$$\{1,2,\ldots,n\} \subset \{j_{k+1},\ldots,j_{k+\Delta}\}, \tag{15}$$

$$\{1,2,\ldots,m\} \subset \{i_{k+1},\ldots,i_{k+\Gamma}\} \tag{16}$$

for every $k \geq 0$.

Result 2. *The MREK algorithm has a linear convergence rate.*

Result 3. *The ACEK algorithm has a sublinear convergence rate.*

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L-functions and Their Applications

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Abstract

In this talk we introduce classical and automorphic L-functions. We give some examples to motivate the general case. Properties of L-functions, conjectures about them and some of their applications are covered.

Introduction

We can introduce L-functions starting from the Riemann zeta function which is the simplest one. An L-function is a type of generating functions of local data associated with either an arithmetic-geometric object or with an automorphic form. There are classical L-functions like Dirichlet series, Hecke L-function attached to a cusp form, L-function of an elliptic curve. An L-function attached to an automorphic form is a represention theoretical treatment of the general case. These are called Langlands' L-functions.

L-functions encode the underlying arithmetical or algebraic structure that is often not easily obtainable by elementary methods, e.g. the classical prime number theorem, class number formula. The fact that the Riemann zeta function has a simple pole at $s = 1$ is an important ingredient of the proof the prime number theorem. Another example is Dirichlet's analytic class number formula which measures the deviation from unique prime factorization in the ring of integers of quadratic number fields. There are many famous conjectures about L-functions. In this talk, we mention some of them.

At the end of the talk we focus on the doubling method. As an example we consider dual reductive pairs of groups over quaternions and obtain integral representations of L-functions attached to a cuspidal representation of these groups. By using doubling method and regularized Siegel-Weil formula location of poles of these L-functions are determined. Also we give some arithmetical results and special values of L-function.

Riemann Zeta Function

Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges absolutely and uniformly in any bounded domain of the complex plane for which $\mathrm{Re}(s) > 1$.

Riemann zeta function has the following properties. General L-functions satisfy similar properties.

- It is analytic for $\mathrm{Re}(s) > 1$ and has meromorphic continuation to entire complex plane,
- The completed zeta function $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ has a functional equation: $\xi(s) = \xi(1-s)$.

  Here gamma function is defined by the Mellin transform of e^{-x}:

  $$\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}\,dx$$

- It has an Euler product:
$$\zeta(s)=\prod_p (1-p^{-s})^{-1}$$
- It has special values: $\zeta(2)=\pi^2/6$. More generally if k is a positive integer $\zeta(2k)$ is a rational multiple of π^{2k}.

Riemann proved the meromorphic continuation of the completed zeta function $\xi(s)$ by obtaining an integral representation. Functional equation follows from this integral representation. There are few methods to obtain meromorphic continuation of L-functions [4].

Classical L-Functions

As an example of classical L-functions we can talk about the standard L-function attached to a cusp form. It is defined as follows: To a cuspform of weight $2k$

$$f(z)=\sum_{n>0} a_n e^{2\pi inz}$$

we attach an L-function

$$L(f,s)=\sum_{n>0}\frac{a_n}{n^s}$$

Similar to the Riemann zeta function, there is an integral representation of the completed L-function.

$$\Lambda(f,s)=(2\pi)^{-s}\Gamma(s)L(f,s)=\int_0^\infty y^s f(iy)\frac{dy}{y}=(-1)^k\Lambda(f,2k-s)$$

Therefore, we have meromorphic continuation and functional equation. Moreover if f is an eigenfunction of the Hecke operators then there is an Euler product:

$$L(f,s)=\prod_p(1-a_p p^{-s}+p^{2k-1-2s})^{-1}=\prod_p(1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$$

Since there are two factors in each p, this is an degree 2 L-function.

Hecke L-functions, Rankin-Selberg L-functions and Artin L-functions are some other important examples of classical L-functions.

Automorphic L-Functions

The Langlands program started 1960's with a series of conjectures connecting seemingly unrelated objects in number theory, algebraic geometry, and the theory of automorphic forms. The connection is given by Langlands' L-functions associated with automorphic representations. There are two kinds of L-functions: motivic L-functions which generalize Artin L-functions and are defined purely arithmetically, and automorphic L-functions, defined by transcendental data. Langlands' reciprocity conjecture claims that every L-function, motivic or automorphic, is equal to a product of L-functions attached to automorphic representations [1].

Let $\mathbf{k}$ a number field and $\mathbf{A}$ its adele ring. Let G be a reductive group. In this talk we consider only general linear groups. Let π be an irreducible automorphic cuspidal representation of $G(\mathbf{A})$. Then we have

$$\pi=\widehat{\bigotimes_v}\pi_v$$

where π_v is an irreducible representation of G_v, almost everywhere locally spherical (unramified).

Let S be the archimedean places together with all finite places v at which π_v is not spherical. Let ${}^LG$ be the L-group of G and r be an algebraic representation of ${}^LG$.

For each $v \notin S$, let $\lambda_v(\pi_v)$ be the conjugacy class in ${}^L G_v$ associated to the restriction of π to G_v. We define the local factors by

$$L(s, \pi_v, r_v) = \frac{1}{\det(1 - r(\lambda(\pi_v))\, q_v^{-s})}$$

where q_v is the order of the residue field of $\mathbf{k}_v$. Here λ is the Satake parameter. Then the (restricted) global L-function is the infinite product

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v)$$

Langlands proved that $L_S(s, \pi, r)$ converges absolutely for $\mathrm{Re}(s)$ sufficently large.

Conjectures

There are many famous conjectures about L-functions. Some important ones can be listed as follows:

1. Analytic conjectures
   (a) Riemann hypothesis: all non-real zeros of $\zeta(s)$ lie on the critical line Re $s = 1$.
   (b) Generalized Riemann hypothesis: similar conjectures for global L-functions.
   (c) Lindelöf hypothesis is about the rate of growth of the Riemann zeta function on the critical line that is implied by the Riemann hypothesis.
   (d) Ramanujan-Petersson conjecture gives a bound for the Fourier coefficients of modular forms or automorphic forms.
2. Algebraic conjectures
   (a) Birch and Swinnerton-Dyer conjecture describes the set of rational solutions to equations defining an elliptic curve. More precisely, it states that the rank of the Mordell-Weil group of an elliptic curve is equal to the order of the zero of the associated L-function $L_E(s)$ at $s = 1$.
   (b) Special values of L-functions: Aim is to generalize formulae such as the Leibniz formula:

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4},$$

   Left-hand side is a special value at $s = 1$ of the Dirichlet L-function for the Gaussian field.
   There are two families of conjectures in the study of special values:
   (i) Beilinson's conjectures: concerned with replacing π in the Leibniz formula by some other transcendental number.
   (ii) Bloch-Kato conjectures: concerned with generalizing the rational factor in the formula by some algebraic construction of a rational number that will represent the ratio of the L-function value to the transcendental factor.

Integral Representations of L-functions

There are few methods to obtain meromorphic continuation of L-functions.

- Jacquet-Langlands: Uses zeta integrals. Representation theoretic treatment of the standard L-function.
- Ranking-Selberg method: Try to get an L-function from a convolution of an Eisenstein series and cuspforms or theta series. Shimura used theta functions; Garrett uses 3 cuspforms to get triple product L-function [2].

- Langlands-Shahidi method: Realize the L-function as a constant term of an Eisenstein series on a larger group. You can only get a fixed and known list of L-functions [4].
- Piatetski-Shapiro and Rallis : Doubling method [3].

Applications

We consider the quaternion dual pair $(O^*(4n), Sp^*(n,n))$. We prove a theorem about the location of possible poles of standard Langlands L-function associated to a given irreducible cuspidal representation of one of these groups [9]. The proof uses the information about the poles of normalized Siegel Eisenstein series [7], the regularized Siegel-Weil formula [12, 13] and the doubling method [3,8].

As an another application we prove arithmeticity results on holomorphic cuspforms on symplectic, unitary, and Hermitian orthogonal groups. More precisely, we establish arithmeticity results on the ratios of Petersson inner products of these cuspforms which are important for studying special values of L-functions. We also show that there is a basis for the space of cuspforms of fixed weight and level consisting of elements with rational-valued Fourier coefficients and we solve the basis problem for holomorphic cusp-forms on these groups. That is, we show that for sufficiently large weights such spaces are spanned by certain types of theta series. [5,9]

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Counting Rational Points and Mild Parametrization

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Abstract

This talk is about counting rational points of bounded height on sets which does not contain any infinite semi-algebraic set, and how mild parametrization is used to obtain an upper bound on the number of these points.

Mild parametrizations, which are parametrizations with some control on the derivatives, were first applied in the realm of diophantine geometry by J. Pila, to obtain results about the density of rational points on the graphs of non-algebraic pfaffian functions. I will try to explain the basic idea of the proof of this result and give a sketch of it.

Furthermore Pila has shown that obtaining mild parametrization results with some conditons satisfied, would be sufficient to establish Wilkie's conjecture. This conjecture is about the density of rational points of sets defined by equation and inequalities involving polynomials and the exponential function. I will also talk about this conjecture and why mild parametrization is considered to be a promising tool in order to establish it.

Introduction

Diophantine geometry studies integer solutions of algebraic equations; one aspect of it is to count the number of rational points on algebraic surfaces. On the other hand, it is also natural to consider the problem on non-algebraic surfaces. In the paper [1], upper bounds for number of integer points on the graphs of transcandental real-analytic functions were established. It was followed by more general result of Pila-Wilkie which states an upper bound for the density of rational points on certain non-algebraic subsets of $\mathbb{R}^n$ which are definable in o-minimal structures [4].

The definition of o-minimal structure will be given in 2 and more information can be found in [7]. The most basic example of an o-minimal structure is the ordered field of reals, $\overline{\mathbb{R}} = (\mathbb{R}, +, -, \times, 0, 1, <)$, and its definable sets are the semi-algebraic sets, see [6]. One can create expansions of $\overline{\mathbb{R}}$ by adding new function symbols to the language. Some examples of such expansions which still stay o-minimal are $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$, where symbols for each restricted analytic function and for the total function $\exp : \mathbb{R} \to \mathbb{R}$ are added to the language respectively.

Work of Pila in [3] shows that, the bound in Pila-Wilkie thorem can not be improved in general. However, Wilkie proposed an improvement to the bound for $\mathbb{R}_{\text{exp}}$, a conjecture which is still open in full generality.

One main ingredient of the proof of Pila-Wilkie theorem is using reparametrizations of concerned sets to see them as images of functions whose derivatives are bounded up to some prescribed order. Pila, in [5], used reparametrizations with functions that have control on all orders of their derivatives, to establish an upper bound for the density of rational points of a specific curves that fit to the setting of Wilkie's conjecture. The significance of this result lies in the improvement of the bound which is the same improvement that Wilkie conjectured. Main objective of this talk is to explain the proof of Pila's result. The key to his method is, rational points of a set $X \subset \mathbb{R}^2$ is contained in few inters of X with plane algebraic curves of suitable degree. I will mostly concentrate on this key part of the proof and draw attention to how it makes use of the reparametrization.

Pila-Wilkie Thorem and Wilkie's conjecture

The sequence $S := (S_n : n \geq 1)$, where each S_n is a collection of subsets of $\mathbb{R}^n$, is called a structure if, for all $n, m \geq 1$, the following conditions are satisfied:

1. S_n is a boolean algebra
2. S_n contains every semi-algebraic subset of $\mathbb{R}^n$
3. if $A \in S_n$ and $B \in S_m$ then ABS_{n+m}
4. if $m \geq n$ and $A \in S_m$ then $\pi(A) \in S_n$, where $\pi : R^m \to R^n$ is projection onto the first n coordinates

If S is a structure and $X \subset R^n$, we say X is definable in S if $X \in S_n$. If S is a structure and, in addition, the boundary of every set in S_1 is finite then S is called an o-minimal structure.

Let $q = \dfrac{a}{b}$ be a rational number. The height of q is defined as, $H(q) := max(|a|, b)$, for $a, b \in \mathbb{Z}$, $b > 0$ and $(a, b) = 1$. For $q = (q_1, \ldots, q_n)$, we define $H(q_1, \ldots, q_n) := max_{1 \leq j \leq n}(H(q_j))$. For a set $X \subset \mathbb{R}^n$ define

$$X(\mathbb{Q}, H) := \{P \in X \cap \mathbb{Q}^n : H(P) \leq H\}.$$

The function $N(X, H) := \#X(\mathbb{Q}, H)$ is called, the density function of X.

Theorem (Pila-Wilkie). *Let $X \subseteq \mathbb{R}^n$ be a definable set in an o-minimal expansion of $\overline{\mathbb{R}}$ which contains no infinite semi-algebraic subset, and let $\varepsilon > 0$. Then there is a constant $c > 0$ such that*

$$N(X, H) \leq cH^{\varepsilon}.$$

[Wilkie's conjecture] Let $X \subseteq \mathbb{R}^n$ be a definable set in $\mathbb{R}_{\exp}$ which does not contain any infinite semi-algebraic set. Then there are constants $c_1, c_2 > 0$ such that, for all $H \geq e$,

$$N(X, H) \leq c_1(logH)^{c_2}.$$

Mild Parametrization

Mild parametrization was introduced by Pila in [5] and it has played a central role in important steps already made towards the conjecture.

Definition (mild function/mild parametrization). *A smooth function $f : (0,1) \to (0,1)$ is said to be mild if there exists $B > 0, C \geq 0$ such that the nth derivative of f is bounded by $n!(Bn^C)^n$. In this case we say that f is B, C-mild. The definition for multivariable functions is analogous. Let $X \subseteq \mathbb{R}^n$. We say that X has mild parametrization if there exist a finite number of mild functions $\phi_1, \ldots \phi_l : (0,1)^{dimX} \to \mathbb{R}^n$ such that union of their images covers X.*

For $\phi : (0,1) \to \mathbb{R}^2$ we will write $M(B, C)$ if $\phi((0,1)) \subseteq [-1,1]^2$ and the coordinate functions f, g of ϕ are both B, C-mild functions.

In [5], Pila proved the following proposition. In this talk, I will try to explain how he made use of mild parametrization to obtain the bound.

[(Pila, 2006). *Let $d > 5$. Let $\phi \in M(B, C)$ with image X and let $H > 1$. Then $X(\mathbb{Q}, H)$ is contained in the union of at most*

$$6BD^C H^{\frac{8}{(d+3)}}$$

algebraic curves of degree d.

Combining the above result with a result of Khovanski about pfaffian functions, Pila concludes that Wilkie's conjecture is true for non-algebraic pfaffian curves. For the definition of pfaffian functions (they contain the function exp) and the result of Khovanski see [2],

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Curves with ordinary singularities

Florin Ambro, IMAR

Abstract

I will discuss the classification of projective curves with at most seminormal singularities, in a similar way to the classification of smooth projective curves.

On the solutions of a Diophantine equation with power sums

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Abstract

This is joint work with Attila Bérczes, István Pink, and Gamze Savaş.
In this work, we are interested in positive integer solutions of the Diophantine equation

$$T_k(x) = y^n \tag{1}$$

where

$$T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k.$$

We first provide upper bounds for n which depend on assertions describing the precise exponents of 2 and 3 appearing in the prime factorization of $T_k(x)$ and on the explicit solution of polynomial exponential congruences. Secondly, we show that the equation (1) has no solutions in positive integer unknowns (x,y,k,n) with $2 \leq x \leq 13$, $y \geq 2$, $k \geq 1$, $n \geq 3$. To prove this, we combine several tools: Baker's method (in particular, sharp bounds for the linear combinations of logarithms of two algebraic numbers), polynomial-exponential congruences and computational methods.

Introduction

Let x and k be positive integers. Write

$$S_k(x) = 1^k + 2^k + ... + x^k$$

for the sum of the $k-th$ powers of the first x positive integers. The Diophantine equation

$$S_k(x) = y^n\,, \tag{2}$$

in unknown positive integers k,n,x,y with $n \geq 2$ has a rich history. In 1875, the classical question of Lucas [12] was whether equation (2) has only the solutions $x = y = 1$ and $x = 24$, $y = 70$ for $(k,n) = (2,2)$. In 1918, Watson [19] solved equation (2) with $(k,n) = (2,2)$. In 1956, Schäffer [16] considered equation (2). He showed, for fixed $k \geq 1$ and $n \geq 2$, that (2) possesses at most finitely many solutions in positive integers x and y, unless

$$(k,n) \in \{(1,2),(3,2),(3,4),(5,2)\} \tag{3}$$

where, in each case, there are infinitely many such solutions. There are several effective and ineffective results concerning equation (3), see the survey paper [8]. Schäffer's conjectured that (3) has the unique non-trivial (i.e. $(x,y) \neq (1,1)$) solution, namely $(k,n,x,y) = (2,2,24,70)$. In 2004, Jacobson, Pintér, Walsh [10] and Bennett, Győry, Pintér [2], proved that the Schäffer's conjecture is true if $2 \leq k \leq 58$, k is even $n = 2$ and $2 \leq k \leq 11$, n is arbitrary, respectively. In 2007, Pintér [4], proved that the equation

$$S_k(x) = y^{2n},\ \text{in positive integers}\ \ x,y,n \text{ with } n > 2 \tag{4}$$

has only the trivial solution $(x,y) = (1,1)$ for odd values of k, with $1 \leq k < 170$.

In 2015, Hajdu [9], proved that Schäffer's conjecture holds under certain assumptions on x, letting all the other parameters free. He also proved that the conjecture is true if $x \equiv 0,3 \pmod 4$ and $x < 25$. The main tools in the proof of this result were the 2-adic valuation of $S_k(x)$ and local methods for polynomial-exponential congruences. Recently Bérczes, Hajdu, Miyazaki and Pink [5], provided all solutions of equation (2) with $1 \leq x < 25$ and $n \geq 3$.
Now we consider the Diophantine equation

$$(x+1)^k + (x+2)^k + ... + (x+d)^k = y^n \tag{5}$$

for fixed positive integers k and d.

In 2013, Zhang and Bai [1], considered the Diophantine equation (5) with $k = 2$. They first proved that all integer solutions of equation (5) such that $n > 1$ and $d = x$ are $(x,y) = (0,0)$, $(x,y,n) = (1,\pm 2,2),(2,\pm 5,2),(24,\pm$ or $(x,y) = (-1,-1)$ with $2 \nmid n$. Secondly, they showed that if $p \equiv \pm 5 \pmod{12}$ is prime, $p \mid d$ and $v_p(d) \not\equiv 0 \pmod n$, then equation (5) has no integer solution (x,y) with $k = 2$. In 2014, the equation

$$(x-1)^k + x^k + (x+1)^k = y^n \quad x,y,n \in \mathbb{Z}, \quad n \geq 2, \tag{6}$$

was solved completely by Zhang [20], for $k = 2,3,4$ and the next year, Bennett, Patel and Siksek [3], extend Zhang's result, completely solving equation (6) in the cases $k = 5$ and $k = 6$. In 2016, Bennett, Patel and Siksek [4], considered the equation (5). They gave the integral solutions to the equation (5) using linear forms in logarithms, sieving and Frey curves where $k = 3$, $2 \leq d \leq 50$, $x \geq 1$ and n is prime.

Let $k \geq 2$ be even, and let r be a non-zero integer. Recently, Patel and Siksek [13], showed that for almost all $d \geq 2$ (in the sense of natural density), the equation

$$x^k + (x+r)^k + ... + (x+(d-1)r)^k = y^n, \quad x,y,n \in \mathbb{Z}, \quad n \geq 2$$

has no solutions. Let $k,l \geq 2$ be fixed integers. More recently, Soydan [5], considered the equation

$$(x+1)^k + (x+2)^k + ... + (lx)^k = y^n, \quad x,y \geq 1, n \in \mathbb{Z}, \quad n \geq 2 \tag{7}$$

in integers. He proved that the equation (7) has only finitely many solutions in positive integers, x,y,k,n where l is even, $n \geq 2$ and $k \neq 1,3$. He also showed that the equation (7) has infinitely many solutions where $n \geq 2$, l is even and $k = 1,3$.

In this work, we are interested in the integer solutions of the equation

$$T_k(x) = y^n \tag{8}$$

where

$$T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k \tag{9}$$

for positive integer k. We provide upper bounds for n and give some results about equation (8).

The main results

Our main results provide upper bounds for the exponent n in equation (8) in terms of 2 and 3-valuations v_2 and v_3 of some functions of x and x,k. Further, on combining Theorem with Baker's method and with a version of the local method (see e.g. [5]), we show that for $2 \leq x \leq 13, k \geq 1, y \geq 2$ and $n \geq 3$ equation (8) has no solutions.

For a prime p and an integer m, let $v_p(m)$ denote the highest exponent v such that $p^v | m$.

Theorem. *([6]) (i) Assume first that $x \equiv 0 \pmod 4$. Then for any solution (k,n,x,y) of equation (8), we get*

$$n \leq \begin{cases} v_2(x)-1, & \text{if } k=1 \text{ or } k \text{ is even,} \\ 2v_2(x)-2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

(ii) *Assume that $x \equiv 1 \pmod 4$ and $k=1$, then for any solution (k,n,x,y) of equation (8), we get $n \leq v_2(3x+1)-1$.*

Suppose next that $x \equiv 1,5 \pmod 8$ and $x \not\equiv 1 \pmod{32}$ with $k \neq 1$. Then for any solution (k,n,x,y) of equation (8), we get

$$n \leq \begin{cases} v_2(7x+1)-1, & \text{if } x \equiv 1 \pmod 8 \text{ and } k{=}2, \\ v_2((5x+3)(3x+1))-2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k=3, \\ v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 3 \text{ is odd,} \\ 1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 2 \text{ is even,} \\ 2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 4 \text{ is even,} \\ 3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 5 \text{ is odd} \\ & \text{or} \\ & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 4 \text{ is even,} \\ 4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 5 \text{ is odd.} \end{cases}$$

(iii) *Suppose now that $x \equiv 0 \pmod 3$ and k is odd or $x \equiv 0,4 \pmod 9$ and $k \geq 2$ is even. Then for any solution (k,n,x,y) of equation (8),*

$$n \leq \begin{cases} v_3(x), & \text{if } x \equiv 0 \pmod 3 \text{ and } k=1, \\ v_3(x)-1, & \text{if } x \equiv 0 \pmod 9 \text{ and } k \geq 2 \text{ is even,} \\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd,} \\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k=3, \\ v_3(2x+1)-1, & \text{if } x \equiv 4 \pmod 9 \text{ and } k \geq 2 \text{ is even.} \end{cases}$$

Theorem. *([6]) Assume that $x \equiv 1,4 \pmod 8$ or $x \equiv 4,5 \pmod 8$. Then Eq. (8) has no solution with $k=1$ or $k \geq 2$ is even, respectively.*

Theorem. *([6]) Consider equation (8) in positive integer unknowns (x,k,y,n) with $2 \leq x \leq 13, k \geq 1, y \geq 2$ and $n \geq 3$. Then equation (8) has no solutions.*

The sketch for the proofs

Using the properties of Bernoulli polynomials, [15] (especially using the fact that Bernoulli polynomials are Appell polynomials), we first give the decomposition of the polynomial $T_k(x)$. Then we obtain formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$ which describe the precise exponents of 2 and 3 appearing in the prime factorization of $T_k(x)$. To prove Theorem and Theorem which give upper bounds for the exponent n in equation (8) in terms of $V_2(T_k(x))$ and $V_3(T_k(x))$, we use congruence properties of the polynomial $S_k(x)$, [17]. Finally, we show that the equation (8) has no solutions in positive integer unknowns (x,k,y,n) with $2 \leq x \leq 13, k \geq 1, y \geq 2$ and $n \geq 3$, i.e. we prove Theorem . To do this, we combine several tools: Baker's method (in particular, Laurent's result [11] on sharp bounds for the linear combinations of logarithms of two algebraic numbers), polynomial-exponential congruences and computational methods (which are supported by MAGMA [7]).

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Computing index of associative algebras

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Abstract

The aim of this work is to compute and discuss the index of associative algebras. We introduce for an associative algebra, in addition to the index, many invariants related to the subspace "Kernel", the minimal and maximal index. We provided also the regular vectors. And finally we give links between the index of associative algebras and Lie algebras by anticommutation.

Introduction

The index theory of Lie algebras was intensively studied by Elashvili see. see ([2], [3], [4]), he considered in particular the case of semi-simple Lie algebras and Frobenius Lie algebras. The index of associative algebras was initiated by Dergachev see ([6]) in order to decompose a finite dimensional associative algebra into subspaces and then provide a classification.

In this work, we remind the general theory of the index of an associative algebra. We introduce the maximal and the minimal notion of the index (respectively to the right and to the left). We also introduce the notion of regular vectors of ordre r. r, is the value of the index. Then, we determine the values and the expressions of the various objects occurring in this theory.

Definitions and properties

Let $\mathscr{A} = (\mathbb{V}, \mu)$ be an associative algebra of finite dimensional, $E = \{e_1, e_2, ..., e_n\}$ be a basis in $\mathbb{V}$, and $f \in \mathbb{V}^*$.

We put $\mathscr{B}_f = f(\mu(x,y))$, and we define $\mathscr{A}_{\setminus f}$ to be the matrix

$$(f(\mu(x,y)))_{ij}.$$

Definition. *We put*

$$\begin{aligned} Ker^L \mathscr{A}_{\setminus f} &= \{x \in \mathscr{A},\ f(\mu(x,y)) = 0,\ \forall y \in \mathscr{A}\} \\ Ker^R \mathscr{A}_{\setminus f} &= \{x \in \mathscr{A},\ f(\mu(y,x)) = 0,\ \forall y \in \mathscr{A}\} \\ Nil_f &= Ker^L \mathscr{A}_{\setminus f} \cap Ker^R \mathscr{A}_{\setminus f}. \end{aligned}$$

[. *If $\mathscr{A}$ is commutative then*

$$Nil_f = Ker^L \mathscr{A}_{\setminus f} = Ker^R \mathscr{A}_{\setminus f}.$$

Definition. *We introduce also the following invariants relative to thesubspace*

$$\dim Ker^L \mathscr{A}_{\setminus f},\ \dim Ker^R \mathscr{A}_{\setminus f}\ \text{and}\ Nil_f$$

$$
\begin{aligned}
m^L_{\mathscr{A}} &= \min_{f\in\mathbb{V}^*}\left\{\dim Ker^L\mathscr{A}_{\backslash f}\right\}\\
M^L_{\mathscr{A}} &= \max_{f\in\mathbb{V}^*}\left\{\dim Ker^L\mathscr{A}_{\backslash f}\right\}\\
m^R_{\mathscr{A}} &= \min_{f\in\mathbb{V}^*}\left\{\dim Ker^R\mathscr{A}_{\backslash f}\right\}\\
M^R_{\mathscr{A}} &= \max_{f\in\mathbb{V}^*}\left\{\dim Ker^R\mathscr{A}_{\backslash f}\right\}\\
\chi_\mu &= \min_{f\in\mathbb{V}^*}\left\{\dim Nil_f\right\}\\
\eta_\mu &= \max_{f\in\mathbb{V}^*}\left\{\dim Nil_f\right\}
\end{aligned}
$$

Definition. *The integer* $\inf_{f\in\mathbb{V}^*}\left\{\dim Nil_f\right\}$ *is called the index of* $\mathscr{A}$. *We say that a vector* $f = \sum_{k\succeq 0} g_k x^*$ *is regular if*

$$\dim Nil_f = \chi_\mu.$$

if $\chi_\mu = r$ *then* f *is called regular of ordre* r.

[. *regular vector of ordre* 0 *is said olso a regular vector. In practice if the index is r, the regular vectors of ordre r are given by the element of dual space* $g_1x_1^* + ... + g_nx_n^*$ *as the monors of ordre* $n-r$ *is of determinant non null.*

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InfoMod: A visual and computational approach to Gauss' binary quadratic forms

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Abstract

InfoMod is a new software and application devoted to the modular group, $PSL_2(\mathbb{Z})$). It has algorithms that deals with the classical correspondences among continued fractions, geodesics on the modular surface and binary quadratic forms. In addition the software implements the recently discovered representation of Gauss' indefinite binary quadratic forms and their classes in terms of certain infinite planar graphs (dessins) called ¸carks. InfoMod illustrates various aspects of these forms, i.e. Gauss' reduction algorithm, the representation problem of forms, ambiguous and reciprocal forms. It can be used as an educational tool, and might be used to explore some new facts about these objects

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Hurwitz class numbers in the theory of automorphic forms and an extension to totally real number fields

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Abstract

The classical Hurwitz class numbers show up in the theory of automorphic forms as Fourier coefficients of a mock modular form of weight 3/2, which is essentially due to the classical fact that they are special values of certain natural L-series. In this talk we recall first of all the classical theory, and then we show how it can be smoothly generalized to a arbitrary totally real number fields.

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The Sato-Tate Conjecture

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Abstract

The recent proof of the Sato-Tate Conjecture for elliptic curves and modular forms is one of the breakthrough results in mathematics by Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron and Taylor. The Sato-Tate Conjecture is a statement about the statistical distribution of certain sequences of numbers. The conjecture was originally stated for elliptic curves in terms of Frobenius angles independently by Mikia Sato (computational) and John Tate (theoretical) around 1960. It is also possible to state the conjecture for modular forms. Let $k \geq 1$ and $f = \sum_{n\geq 1} a(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication. Then the Sato-Tate Conjecture says that the numbers $\frac{a(p)}{2p^{k-1/2}}$ are equidistributed in $[-1,1]$ with respect to a certain measure when p runs through the primes not dividing N. In this talk, we will consider the Sato-Tate Conjecture for modular forms with its history and consequences.

Introduction: Modular Forms and Elliptic Curves

Modular forms are holomorphic functions on the upper half plane that satisfy fundamental symmetry and growth conditions. Roughly speaking, you can think about them as functions that have a certain transformation behaviour under Mobius transformations.

Let $SL(2,\mathbb{Z})$ be the modular group, namely,

$$\Gamma = SL(2,\mathbb{Z}) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = 1 \}.$$

Let f be a complex-valued function on the upper half plane

$$\mathfrak{U} := \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$$

satisfying the following three conditions:

1. f is a holomorphic on $\mathfrak{U}$,
2. for any $z \in \mathfrak{U}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

   where k is a positive integer,
3. f is holomorphic at ∞.

Then f is called a *modular form of weight k for the modular group* Γ.

A modular form f that vanishes at ∞ is called a *cusp form.* A standard reference for modular forms is [7].

It is clear that $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are generators of the modular group and $T(z) := z+1$ and $S := -1/z$, so we have

$$f(-1/z) = z^k f(z)$$

and

$$f(z+1) = f(z)$$

One can define modular forms for the subgroups of Γ, for instance for the most famous *congruence subgroup*, namely,

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0(\mathrm{mod}N)\}.$$

In this case, we add to the definition "level N". It is clear that a modular form for Γ is of level 1.

Since modular forms are periodic they are perfect number theoretic objects, i.e. f admits a Fourier expansion

$$f(z) = \sum_{n\geq 0} a_n q^n$$

where $q = e^{2\pi iz}$, $z \in \mathfrak{U}$.

The space of modular forms is denoted by $M_k(\Gamma)$ and cusp forms by $S_k(\Gamma)$ and they are finite dimensional vector space over $\mathbb{C}$. We can easily compute dimensions of these with a certain formula for $k \geq 2$. So modular forms are computation-friendly! Beside many applications in several branches of mathematics and even in physics, here we restrict our attention to their applications in number theory.

[. *The simplest example for the modular forms is* the Ramanujan tau function*:*

$$\Delta(z) := \sum_{n\geq 1} \tau(n) q^n = q \prod_{n\geq 1} (1-q^n)^{24} = \eta(z)^{24}$$

which is a modular form of weight 12 *for* Γ *and level* 1.

We have

$$\tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472, \tau(5) = 4830, \tau(6) = -6048, ...$$

Values of tau functions have interesting relations, which were observed by Ramanujan, one of them is trailer for the following lines of the text:

Theorem. *For all primes p, one has*

$$|\tau(p)| \leq 2p^{11/2}$$

Actually, more is true! We will give Ramanujan-Petersson bound (Deligne 1974, in [3]) later on below.

Another interesting example for modular forms is Eisenstein series.

Definition. *An Eisenstein series with half-period ratio z and index k is defined by*

$$G_k(z) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m+nz)^k}$$

where m and n are not simultaneously equal to 0*,* $z \in \mathfrak{U}$ *and* $k > 2$ *an integer.*

Eisenstein series have the following nice property:

Theorem. *[7] For $k \geq 2$, $G_{2k} \in M_{2k}(\Gamma)$.*

To handle modular forms efficiently, we need *Hecke operators*, which are a certain kind of averaging operators that play an important role in the structure of vector spaces of modular forms and more general automorphic representations.

Definition. *For a fixed integer k and any positive integer n,* n_th Hecke operator *is denoted by T_n and it is defined on the set $M_k(\Gamma)$ as $T_n : M_k(\Gamma) \to M_k(\Gamma)$ by*

$$(T_n f)(z) := n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{az+bd}{d^2}\right)$$

Note that T_n preserves the space of cusp forms and they have the properties $T_m \circ T_n = T_{mn}$ if $gcd(m,n) = 1$ and for prime p, $T_p \circ T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$.

We have the following special modular forms:

Definition. *An* eigenform *is a modular form which is an eigenvector for all Hecke operators T_m for $m = 1, 2, \ldots$.*

Definition. *An eigenform is said to be* normalized *when scaled so that the $q-$coefficient in its Fourier series is one:*

$$f = a_0 + q + \sum_{n=2}^{\infty} a_n q^n$$

Note that eigenform means simultaneous Hecke eigenform with Γ and existence of eigenforms is a nontrivial result but it comes from the fact that *Hecke algebra* is commutative.

It is clear that coefficients of normalized eigenforms are exactly eigenvalues of Hecke operators, i.e.

$$T_n f = a_n f.$$

Definition. *A* newform *is a cuspform of level N that does not come from strictly lower weight.*

Definition. *The L-function of a cusp form f is denoted by $L(s,f)$ and defined as*

$$L(s,f) := \sum_{n=1}^{\infty} = \prod_{p|N}(1 - \tfrac{a_p}{p^s})^{-1} . \prod_{p\nmid N}(1 - \tfrac{a_p}{p^s} + \tfrac{1}{p^{2s+1-k}})^{-1}$$

Definition. *A newform $f = \sum_{n=1}^{\infty} a_n q^n$ of level N and weight k has complex multiplication if there is a quadratic imaginary field K such that $a_p = 0$ as soon as p is a prime which is inert in K. Otherwise it is defined that f is a newform without complex multiplication.*

We resume with *elliptic curves*:

Let $\mathbb{K}$ be a field such that $char(\mathbb{K}) \neq 2, 3$. For fixed $A, B \in \mathbb{Z}$, let us consider the following set:

$$E/K := \{(x,y) \in K : y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

where $\Delta := 4A^3 + 27B^2 \neq 0$. We can define an addition rule on E/K for $P, Q \in E/K$ as

Then with this operation, E/K forms an abelian group. Note that we need ∞ for technical reasons in group law! For the details, the reader is referred to [5].

Let p a prime such that $(p, \Delta) = 1$. Then $E/\mathbb{F}_p$ forms an elliptic curve after reduction. In this case we say *E has a good reduction at p* and p is called *good prime for E*. Otherwise we say *E has a bad reduction at p* and p is called *bad prime for E*.

Suppose that E has a good reduction at p. Then let us denote the number of points on $E/\mathbb{F}_p$ with N_p, i.e.,

$$N_p := \#\{(x,y) \in \mathbb{F}_p : y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

Definition. *$L-$function of an elliptic curve $E/\mathbb{Q}$ is denoted by $L(E,s)$ and defined by*

$$L(E,s) := \prod_{p\text{good}} \frac{1}{1-a(p)p^{-s}+p^{1-2s}}$$

where $a(p) := p+1-N_p$.

The following theorem is called *Modularity Theorem* and it gives a bridge between modular forms and elliptic curves which leads to the proof of the Fermat's Last Theorem due to Wiles (1995) with contributions by Taniyama, Shimura, Frey, Serre, Ribet, Taylor and others:

Theorem. *(Wiles 1995) [11] Let E be an elliptic curve over $\mathbb{Q}$. Then $L(E,s) = L(s,f)$ for some normalized eigenform of weight 2 for $\Gamma_0(N)$ where N is the "conductor" of E.*

Sato-Tate Conjecture

After a long warm-up session, we are ready to state the Sate-Tate Conjecture. Thanks to the Modularity Theorem, we can formulate the conjecture in both direction, namely Sato-Tate Conjecture for elliptic curves and Sato-Tate Conjecture for modular forms. Let us start with the first.

Theorem. *(Harris et al 2010) [4] Let E be an elliptic curve without complex multiplication and p is prime number. Define θ_p by $p+1-\#E(\mathbb{F}_p) = 2\sqrt{p}\cos\theta_p$ where $0 \leq \theta_p \leq \pi$. Then for $0 \leq \alpha < \beta \leq \pi$*

$$\lim_{N\to\infty} \frac{\{p : p \leq N, \alpha \leq \theta_p \leq \beta\}}{\{p : p \leq N\}} = \frac{2}{\pi}\int_\alpha^\beta \sin^2\theta d\theta$$

.

Before giving the Sato-Tate Conjecture for modular forms, we need more preparation. Let $f = \sum_{n\geq 1} a(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication and p a prime number. Then Ramanujan-Petersson bound gives

$$\mid a(p) \mid \leq 2p^{k-1/2}$$

. Using this bound we can make a normalisation on the Fourier coefficients of this Hecke eigenform, namely,

$$b(p) := \frac{a(p)}{2p^{k-1/2}} \in [-1,1]$$

.

One defines the *Sato-Tate measure* μ to be the probability measure on the interval $[-1,1]$ given by $\frac{2}{\pi}\int_a^b \sqrt{1-t^2}dt$.

So we are ready to state the Sato-Tate conjecture for modular forms:

Theorem. *(Barnet-Lamb et al 2011) [1] Let $k \geq 1$ and let $f = \sum_{n\geq 1} a(n)q^n$ be a normalised cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ without complex multiplication. Then the numbers $b(p) = \frac{a(p)}{p^{k-1/2}}$ are μ-equidistributed in $[-1,1]$, when p runs through the primes not dividing N.*

As a corollary of this result, we have the following:

Let $[a,b] \subseteq [-1,1]$ be a subinterval and $S_{[a,b]} := \{p : (p,N) = 1, b(p) \in [a,b]\}$. Then $S_{[a,b]}$ has natural density equal to $\frac{2}{\pi}\int_a^b \sqrt{1-t^2}dt$.

Following graph is drawn by William Stein and it illustrates the Sato-Tate Conjecture for the modular form Δ for $p < 1.000.000$.

The Sato-Tate Conjecture is also proven for Hilbert modular forms which can be considered as automorphic forms generalising modular forms.

Consequences of the Sato-Tate Conjecture

Sato-Tate conjecture has striking consequences, here we will introduce just a small piece. In [2] (later formally in [6]), Bruinier and Kohnen suggested a problem on sign equidistribution of Fourier coefficients of half-integral weight modular forms. This question is based on a celebrated theorem of Waldspurger on L-functions and Fourier coefficients. Under some certain conditions, they claim that the proportion of positive coefficients to negative coefficients is equal among non-zero coefficients. It is an interesting but hard question with the current tools in "half-integral weight modular forms" (we skip some definitions here, due to limited space in booklet). We can consider this problem on some certain subsets of the coefficients. There is an explicit connection with half-integral and integral weight modular forms, called "Shimura correspondance". With this setting, Bruinier-Kohnen sign equidistribution problem becomes a Sato-Tate problem. Hence, we have the following:

Theorem. *(I, Wiese 2013) [5] Let* $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{k+1/2}(N,\chi)$ *be a non-zero cuspidal Hecke eigenform where* χ *at most quadratic character and* N *is an integer such that* $4 \mid N$ *and suppose that* F_t *is the Hecke eigenform which has no complex multiplication and which corresponds to* f *via the Shimura correspondance.*

Define the set of primes

$$\mathbb{P}_{>0} := \{p \in \mathbb{P} | a(tp^2) > 0\}$$

and similarly $\mathbb{P}_{>0}$, $\mathbb{P}_{<0}$, $\mathbb{P}_{\geq 0}$, $\mathbb{P}_{\leq 0}$ *(depending on* f *and* t*).*

Then the sets $\mathbb{P}_{>0}$, $\mathbb{P}_{<0}$, $\mathbb{P}_{\geq 0}$, $\mathbb{P}_{\leq 0}$ *have natural density* $1/2$ *and the set* $\mathbb{P}_{=0}$ *has natural density* 0.

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Dynamics on Flat Surfaces with Finite Holonomy Groups

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Abstract

We prove that closed flat surfaces with finite holonomy groups can be covered by very flat surfaces, and then we state several some immediate consequences of this fact.

Introduction

A surface is called flat if it has a metric obtained by gluing Euclidean triangles along their edges by isometries. Such a metric is also called flat. These metrics are path metrics; they are obtained from lengths of the curves in these triangles. Therefore these surfaces have nice properties: there are length minimizing geodesics between any given two points on them, each free homotopy class of paths on them contains a geodesic representative. See [1], [2] for more information about geometry of these surfaces.

In this paper we will discuss ergodicity of geodesic flow on the flat surfaces with finite holonomy groups. Indeed we will reduce study of these surfaces to certain surfaces which are called very flat.

A very flat surface is a closed, orientable surface with a flat metric having trivial holonomy group. These surfaces may be thought as Riemann surfaces with fixed holomorphic abelian differentials. Also since holonomy of a very flat surface is trivial there is a well-defined notion of direction on each of such surfaces. Now we state a main theorem about ergodicity of geodesic flows on these surfaces. See [6].

[**Kerckhoff, Masur, Smillie] For almost all directions, directional flow on a very flat surface is uniquely ergodic.** We will use this theorem to study geodesic flow on flat surfaces having finite holonomy groups. See [3], [5] and [4] for introduction to the dynamics on very flat surfaces.

Flat Maps and Really Flat Surfaces

In this section we introduce the surfaces and the maps that we want to study. Let S and S^* be two flat surfaces. A map $\psi : S^* \to S$ is called flat if it is a branched covering and local isometry outside of its branched locus.

Now we give examples of flat maps. Take a flat surface S and a topological branched cover $\psi : S^* \to S$ of it. This map induces a flat metric on S^* and ψ is flat with respect to this metric. Now we give a concrete example of this construction. We know that there is a branched cover of degree two from a torus to a sphere which is branched over four points. If we put a flat structure on the sphere with singularities at branched points, we get a flat map between a flat torus and a tetrahedron. Note that that each flat torus is a degree two flat cover of a tetrahedron.

It is clear that if ramification index of ψ at x is m, then $\theta(x) = m\theta(\psi(x))$, where $\theta(x)$ is the angle at x. Now we rename the surfaces that we want to study. By double of a surface S with boundary, we mean the surface obtained by gluing two copies of S along their boundary.

- Let S be a flat surface without boundary. S is called really flat if its holonomy group is finite.
- Let S be a flat surface with boundary. S is called really flat if its double is really flat.

Holonomy Representation of Really Flat Surfaces

Let S be a really flat surface and $\mathbf{s}$ be the set of singular points of S. Since the metric, or the connection, on $S \setminus \mathbf{s}$ is flat, for each $y \in S \setminus \mathbf{s}$ the holonomy representation

$$hol : \Pi_1(S - \mathbf{s}, \mathbf{y}) \to \mathbf{T_y}$$

is well defined, where $\mathbf{T_y}$ is group of rotations of unit circle at tangent space of S at $\mathbf{y}$.

Observe that one can find a polygonal loop in each homotopy class of loops based at $\mathbf{y}$. Since holonomy of a polygonal loop given by angles at vertices of the loop, it follows that flat maps preserve holonomy. We state this fact in a more formal manner.

[. *Let $\psi : S^* \to S$ be a flat map between closed oriented surfaces. Let $\mathbf{b}$ the set of branched points of S and $\mathbf{s}(S)$ is the set of singular points of S. Let l be a loop in $S - \psi^{-1}(\mathbf{b} \cup \mathbf{s}(S))$ based at y. Denote homotopy class of l by $[l]$. Then*

$$hol([l]) = hol(\psi_*([l])).$$

The Main Theorem and Its Applications

In this section we introduce the main theorem of the present paper and give some applications of it.

Let S be a closed, oriented really flat surface. There is a flat covering $\psi : S^* \to S$ so that S^* is very flat.

Proof. For each $x \in S$, let $\theta(x) = 2\pi \frac{k(x)}{l(x)}$, where $k(x)$ and $l(x)$ are relatively prime positive integers. Let $\mathbf{s} = \{x_1, \ldots x_n\}$ be the set of the singular points of S. Also let

$$D = \sum_{i=1}^{n} l(x_i) x_i$$

be a formal divisor on S. Observe that the holonomy representation of S induces a map

$$\bar{hol} : \ \Pi_1^{orb}(S, D) \to \mathbf{T_y},$$

where y is any base point. Let $\mathbf{K}$ be the kernel of $\bar{hol}$. Let $\psi : S^* \to S$ be the branched cover corresponding to $\mathbf{K}$. Lemma implies that S^* is very flat. □

If we are given a rational polygon then we can obtain a very flat surface from several copies of it by Katok-Zemliyakov construction. See [7], [6]. This means that the double of a rational polygon can be covered by a very flat surface. Also, we can obtain a very flat cover of a double of a rational polygon by following the proof of the above theorem. It follows that the two very flat surfaces (and flat maps) obtained by these methods are same. Therefore the method introduced above is a natural generalization of Katok-Zemliyakov construction.

Now we state two immediate applications of the above theorem.

1. Since geodesic flow on a very flat surface for almost all directions is ergodic and every closed really flat surface can be covered by a very flat surface, it follows that flow of a generic geodesic on a really flat surface is ergodic.

2. Since double of a really flat surface is really flat, the item above implies that flow of a generic billiard trajectory on a really flat surface with boundary is ergodic.

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Some Inverse Problems in Computer Algebra

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Abstract

Computer algebra has become a tool for facilitating combinatorial, number theoretic, algebraic, geometric ... proofs. Computers can even produce fully-fledged proofs, especially in the theory of special functions. We will first review some major developments in computer algebra in the last century, then introduce a type of inverse problem to the current theory. As time allows, we will go over concrete examples.

Computer Algebra and Special Functions

Computer algebra is used in many branches of mathematics for facilitating or even producing proofs. We will zero in our attention on identity proving in the context of hypergeometric and q-hypergeometric functions.

A hypergeometric function is a series $\sum_{n\geq 0} \alpha_n$ where the ratio $\frac{\alpha_{n+1}}{\alpha_n}$ is a rational function of n. A q-hypergeometric, or basic hypergeometric, function is a series where the aforementioned ratio is a rational function of q^n. Both are also known as special functions. Other variables or parameters may be involved. The exponential function, trigonometric functions, rational functions, Bessel functions, Chebishev polynomials, etc. are all hypergeometric functions. As a warm-up example, take the series for the sine function.

$$\sin z = \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Here,

$$\frac{\alpha_{n+1}}{\alpha_n} = \cdots = -\frac{z^2}{(2n+2)(2n+3)}.$$

This is indeed a rational function of n with the extra variable z.

Convergence is not an immediate concern, as we can either regard the series as formal series, or impose explicit conditions which make the series absolutely convergent.

The theory of automated identity proving, or WZ-theory, is developed in the legendary book $A = B$ of Petkovšek, Wilf and Zeilberger with extensive historical or otherwise references [5]. The primary questions in the book are the following. Consider the series

$$f(n) = \sum_k F(n,k)$$

where k may run over all integers, non-negative integers or $m \leq k \leq M$; and $F(n,k)$ is a hypergeometric term in both parameters n and k. Can we find a closed formula for $f(n)$? If not, can we prove that none exists? WZ-theory answers these questions affirmatively in a very broad setting. For example, we are now able to prove by computer that

$$\sum_{n\geq 0} \binom{n}{k} = 2^n, \qquad \sum_{n\geq 0} \binom{n}{k}^2 = \binom{2n}{n},$$

and that

$$\text{neither} \sum_{n=0}^{m} \binom{n}{k} \qquad \text{nor} \sum_{n\geq 0} \binom{n}{k}^3$$

have closed formulas. We can also verify or falsify proposed identities between hypergeometric functions purely by computer algebra. It is important that falsification does not involve finding a counterexample.

The starting point of WZ-theory is finding a functional equation for $f(n)$. Then the search for a closed formula or proof that none exists depends on analyzing that functional equation.

Rogers-Ramanujan Identities and their Generalizations

An integer partition of $n \in \mathbb{N}$ is an unordered sum of positive integers $n = \lambda_1 + \cdots + \lambda_l$.

$$9 = 4+4+1 = 5+3+1$$

are integer partitions of 9 into 3 parts. Incidentally, the first one involves parts that are 1 or 4 modulo 5, and the second one involves parts that are neither repeated nor adjacent (consecutive). In fact, there are 5 partitions of 9 that involve parts that are 1 or 4 modulo 5.

$$9 = 9 = 6+1+1+1 = 4+4+1 = 4+1+\cdots+1 = 1+\cdots+1.$$

There are also 5 partitions of 9 into distinct (not repeated) and non-consecutive parts.

$$9 = 9 = 8+1 = 7+2 = 6+3 = 5+3+1.$$

The first of the celebrated Rogers-Ramanujan identities asserts that this equinumerity holds for all non-negative integers [4].

In q-series form, the said identity is

$$1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

The integer partitions literature is abundant with generalizations of Rogers-Ramanujan generalizations. For a good start, the reader is referred to the references in chapter 7 of [1]. There are also computer-aided proofs of these identities [5].

An Inverse Problem

Recently, the author developed a computational method to produce and at the same time prove similar partition identities [2, 3]. The conventional way of proving many partition identities is finding a system of functional equations first. Those equations are satisfied by the generating function of some class of partitions. Then, a twist of a well-known series (almost always Andrews' J-function [1, Ch. 7]) is shown to satisfy the same set of equations, hence an identity is obtained. The proposed method is linear in the sense that once the functional equations are written, their solutions are constructed from scratch, so the second parts of proofs are not independent computations.

An issue with the construction is that it takes too long by hand. The nature of computations indicate that most of the process can be automated. A computer program which takes descriptions of various partitions as inputs, and producing series as generating functions will be highly valuable. Then, a whole bunch of Rogers-Ramanujan type theorems may be obtained effortlessly.

The inverse problem, therefore, is producing series that satisfy the given system of functional equations.

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Irreducible recurrence, ergodicity, and extremality of invariant measures for resolvents

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Abstract

We analyze the transience, recurrence, and irreducibility properties of general sub-Markovian resolvents of kernels and their duals, with respect to a fixed sub-invariant measure m. We prove the equivalence between the m-irreducible recurrence of the resolvent and the extremality of m in the set of all invariant measures, and we apply this result to the extremality of Gibbs states. The talk is essentially based on the joint work [1] with Iulian Cîmpean and Michael Röckner.

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Toric Codes and Lattice Ideals

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Abstract

Let X be a complete n-dimensional simplicial toric variety over a finite field with homogeneous coordinate ring S and TX be its maximal torus. We show that vanishing ideals of certain subsets of TX is a lattice ideal. We identify the lattice corresponding to a degenerate torus in X and completely characterize when its lattice ideal is a complete intersection.

Introduction

Let X be a complete simplicial toric variety of dimension n over the field $=_q$, corresponding to a fan Σ and $T_X \cong (^*)^n$ be its maximal torus. Denote by $\rho_1,\dots,\rho_r$ the rays in Σ and $\mathbf{v}_1,\dots,\mathbf{v}_r \in \mathbb{Z}^n$ the corresponding primitive lattice vectors generating them. Given a vector $\mathbf{u} \in \mathbb{Z}^m$ we use $x^{\mathbf{u}}$ to denote the Laurent monomial $\mathbf{x}^{\mathbf{u}} = x_1^{u_1}\dots x_m^{u_m}$. We also use $[m]$ to denote the set $\{1,\dots,m\}$ for any positive integer $m \geq 1$. Recall the following dual exact sequences:

$$\mathfrak{P} : 0[r]\mathbb{Z}^n[r]^{\phi}\mathbb{Z}^r[r]^{\beta}[r]0,$$

where ϕ is the matrix with rows $\mathbf{v}_1,\dots,\mathbf{v}_r$, and

$$\mathfrak{P}^* : 1[r]G[r]^{ir}[r]^{\pi}T_X[r]1,$$

where $\pi : (t_1,\dots,t_r) \mapsto [\mathbf{t}^{\mathbf{u}_1} : \dots : \mathbf{t}^{\mathbf{u}_n}]$, with $\mathbf{u}_1,\dots,\mathbf{u}_n$ being the columns of ϕ.

Let $S = [x_1,\dots,x_r] = \bigoplus_{\alpha\in} S_\alpha$ be the homogeneous coordinate or Cox ring of X, multigraded by $\cong$ $\mathrm{Cl}(X)$ via $\beta_j := \deg(x_j) := \beta(e_j)$, where e_j is the standart basis element of $\mathbb{Z}^r$ for each $j \in [r]$. The irrelevant ideal is $B = \langle x^{\hat{\sigma}} \ : \ \sigma \in \Sigma \rangle$, where $x^{\hat{\sigma}} = \Pi_{\rho_i \notin \sigma} x_i$. Thus, $T_X \cong^r /G$ and $X \cong (^r \backslash V(B))/G$ as a geometric quotient. The homogeneous polynomials of S are supported in the semigroup $\mathbb{N}\beta$ generated by $\beta_1,\dots,\beta_r$, i.e. $\dim S_\alpha = 0$ when $\alpha \notin \mathbb{N}\beta$.

Next, we recall evaluation codes defined on subsets $Y = \{p_1,\dots,p_N\}$ of the torus T_X. Fix a degree $\alpha \in \mathbb{N}\beta$ and a monomial $F_0 = \mathbf{x}^{\phi(\mathbf{m}_0)+\mathbf{a}} \in S_\alpha$, where $\mathbf{m}_0 \in \mathbb{Z}^n$, $\mathbf{a}$ is any element of $\mathbb{Z}^r$ with $\deg(\mathbf{a}) = \alpha$, and ϕ as in the exact sequence $\mathfrak{P}$. This defines the *evaluation map*

$$\mathrm{ev} : S_\alpha \to_q^N, \quad F \mapsto \left(\frac{F(p_1)}{F_0(p_1)},\dots,\frac{F(p_N)}{F_0(p_N)}\right). \tag{1}$$

The image $_{\alpha,Y} = \mathrm{ev}_Y(S_\alpha)$ is a linear code, called the *generalized toric code*. The block-length N, the dimension $k = \dim_q(_{\alpha,Y})$, and the minimum distance $d = d()$ are three basic parameters of $_{\alpha,Y}$. Minimum distance is the minimum of the number of nonzero components of nonzero vectors in $_{\alpha,Y}$. Toric codes was introduced for the first time by Hansen in [2, 3] for the special case of $Y = T_X$. Clearly, the block-length of $_{\alpha,Y}$ equals $N = |T_X| = (q-1)^n$ in this case. But it is not this easy in the general case. An algebraic way to compute the dimension and length of a generalized toric code is given in [5]. This method is based on the observation that the kernel of the evaluation map above is determined by the vanishing ideal of Y defined as follows. For $Y \subset X$, we define the vanishing ideal $I(Y)$ of Y to be the ideal generated by homogeneous polynomials vanishing on Y. $I(Y)$ is a *complete intersection* if it is generated by a regular sequence of homogeneous polynomials $F_1,\dots,F_k \in S$ where k is the codimension of Y in X. When the vanishing ideal I_Y is a complete intersection, bounds on the minimum distance of $_{\alpha,Y}$ is provided in 1001[6]. Motivated by these results we study vanishing ideals of special subsets of the torus T_X and characterize when they are complete intersections.

Lattice Ideals as Vanishing Ideals

In this section, we show that vanishing ideals of certain subsets of T_X are lattice ideals. By a lattice L we mean a finitely generated free Abelian group. Recall that every vector in $\mathbb{Z}^r$ is written as $\mathbf{m} = \mathbf{m}^+ - \mathbf{m}^-$, where $\mathbf{m}^+, \mathbf{m}^- \in \mathbb{N}^r$. Letting $F_{\mathbf{m}} = \mathbf{x}^{\mathbf{m}^+} - \mathbf{x}^{\mathbf{m}^-}$, the lattice ideal I_L is the binomial ideal generated by special binomials $F_{\mathbf{m}}$ arising from the lattice $L \subset \mathbb{Z}^r$. So, $I_L = \langle F_{\mathbf{m}} \,|\, \mathbf{m} \in L \rangle$. Let $[p] := \pi(p) = [p_1 : \cdots : p_r]$ be a point in X and let $I([p])$ be the vanishing ideal of $[p]$. We use $[1]$ to denote $[1 : \cdots : 1]$. If $[p],[p'] \in X$ then $[p]\cdot[p'] := [pp']$ is well-defined element of $X \cup [V(B)]$, where $[V(B)]$ denotes the set of all $[p]$ for $p \in V(B)$. The set $X \cup [V(B)]$ is a monoid with identity $[1]$ with respect to this coordinatewise multiplication operation.

If $Y \subset T_X$ is a submonoid, then the vanishing ideal $I(Y)$ is a lattice ideal.

Degenerate Tori

The subset $Y_A = \{[t_1^{a_1} : \cdots : t_r^{a_r}] \,:\, t_i \in^*\}$ of the torus T_X is called a **degenerate torus**. If $^* = \langle \eta \rangle$, every $t_i \in^*$ is of the form $t_i = \eta^{s_i}$, for some $0 \le s_i \le q-2$. Let $d_i = |\eta^{a_i}|$ and $D = diag(d_1,\dots,d_r)$ be the matrix defining an automorphism of $\mathbb{Z}^r$. As Y_A is a monoid in T_X, $I(Y_A)$ is a lattice ideal. We determine the corresponding lattice in this section. Given an integer matrix B, let $L_B = \mathbb{Z}^r \cap \ker B$.

If $Y = Y_A$ then $I(Y) = I_L$ for $L = D(L_{\beta D})$.

1 Complete Intersections

We characterize when the vanishing ideals are complete intersections using mixed dominating matrices we define now. If each column of a matrix has both a positive and a negative entry we say that it is *mixed*. If it does not have a square mixed submatrix, then it is called *dominating*.

[[9]] Let L be a non-zero sublattice of $\mathbb{Z}^r$ such that $L \cap \mathbb{N}^r = \{0\}$ and Γ be a matrix whose columns constitute a basis of L. Then I_L is a complete intersection iff Γ is mixed dominating.

Using Theorem 1, we prove the following.

$I(Y_A)$ is a complete intersection iff so is the toric ideal $I_{L_{\beta D}}$. A minimal generating system of binomials for $I(Y_A)$ is obtained from that of $I_{L_{\beta D}}$ by replacing x_i with $x_i^{d_i}$.

We have the following:
(**i**) if $Y = \{[1]\}$ then $I(Y) = I_{L_\beta}$,
(**ii**) if $Y = T_X$ then $I(Y) = I_L$, for $L = (q-1)L_\beta$,
(**iii**) $I(T_X)$ is a complete intersection iff so is I_{L_β}, which is independent of q.

These result generalize some work of [1] from weighted projective spaces to a general toric variety. Using the matrix ϕ defined by the fan Σ and the result presented in this section one can easily check whether the vanishing ideal of T_X is a complete intersection.

Acknowledgments

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Refined Estimates in the Kadison-Singer Problem

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Abstract

The Kadison-Singer problem, posed in 1959, and solved by Adam Marcus, Daniel Spielman and Nikhil Srivastava in 2013, can be seen as basic question about the existence of small norm block compressions (called pavings) of finite matrices. With a view to applications in functional analysis, an important question that was subsequently asked, was if one could jointly pave a tuple of matrices. In this note, we outline a proof of this, obtained jointly with Nikhil Srivastava.

Introduction

In 2013, Adam Marcus, Dan Spielman and Nikhil Srivastava [MSS15], proved the following theorem, previously known as Joel Anderson's paving conjecture [And79], and which was known to imply a positive solution to the Kadison-Singer problem.

Theorem (Marcus, Spielman, Srivastava, 2013)**.** *For any $\varepsilon > 0$ and any zero diagonal hermitian contraction $A \in M_n(\mathbb{C})$, there is a collection of orthogonal diagonal projections $P_1, \cdots, P_r$ where $r \leq C\varepsilon^{-2}$ such that,*

$$P_1 + \cdots + P_r = I, \qquad \lambda_{max}(P_k A P_k) < \varepsilon, \quad k \in [r].$$

Given a not necessarily hermitian zero diagonal matrix A, the above theorem can be applied to the four tuple $(\pm\mathrm{Re}(A), \pm\mathrm{Im}(A))$ to yield,

Theorem (Marcus, Spielman, Srivastava, 2013)**.** *For any $\varepsilon > 0$ and any zero diagonal hermitian contraction $A \in M_n(\mathbb{C})$, there is a collection of orthogonal diagonal projections $P_1, \cdots, P_r$ where $r \leq C\varepsilon^{-8}$ such that,*

$$P_1 + \cdots + P_r = I, \qquad ||P_k A P_k|| < \varepsilon, \quad k \in [r].$$

It was conjectured [PV15], that the asymptotic behaviour of the required size of the paving, $O(\varepsilon^{-8})$, is suboptimal and that one should instead have the same dependance, namely $O(\varepsilon^{-2})$, as one has in theorem (1). In recent work, joint with Nikhil Srivastava [RS17], we settled this by showing that one may a single paving that works for a tuple of matrices,

Theorem (R, Srivastava, 2017)**.** *Given zero diagonal Hermitian contractions $A^{(1)}, \cdots, A^{(k)} \in M_n(\mathbb{C})$ and $\varepsilon > 0$, there exists a paving $X_1 \cdots X_r$ where $r \leq 18k\varepsilon^{-2}$ such that,*

$$\lambda_{max}(P_{X_i} A_j P_{X_i}) \leq \varepsilon, \quad i \in [r],\ j \in [k].$$

As a corollary, we are able to improve the $O(\varepsilon^{-8})$ bound in theorem (1) to the conjectured $O(\varepsilon^{-2})$. We also show that our bound is tight in both k and ε. Further, a simple application of our theorem allows us to prove the following, which is an improvement of a well known recent theorem of Johnson, Ozawa and Scehchtman [JOS13],

Theorem (R, Srivastava, 2017)**.** *If $A \in M_n(\mathbb{C})$ is zero trace, then there we may write $A = [B, C]$ such that,*

$$||B||\,||C|| \leq 300 e^{9\sqrt{\log(n)}} ||A||.$$

Summary of results

Our proof is an adaptation of the techniques of [MSS15], which proceeds in three steps. First, a one-sided paving is obtained via the method of interlacing families of polynomials (see e.g. [MSS14] for an overview), which relies on three facts:

(I) The largest eigenvalue of a Hermitian matrix is equal to the largest root of its characteristic polynomial.

(II) The characteristic polynomials of the matrices:

$$\sum_{i\leq r} P_{X_i} M P_{X_i}, \quad X_1 \cup \ldots \cup X_r = [n]$$

form an *interlacing family*, which implies in particular, that their average is real-rooted and that there exists a polynomial in the family whose largest root is at most that of the sum.

(III) The largest root of the average polynomial can be bounded using a "barrier function argument".

This method is only able to control the largest eigenvalue of a matrix, and therefore can only give one-sided bounds. The suboptimal exponential dependence on k for simultaneously paving k matrices arises from sequentially applying the one-sided result in a black box way k times.

Our main contribution is a technique for simultaneously carrying out the interlacing argument "in parallel" for k matrices, losing only a factor of k in the process. The key idea is to replace the determinant of a single matrix by a *mixed determinant* of k matrices.

Definition. *Given a k tuple of matrices $\mathbf{A} = (A^{(1)}, \cdots, A^{(k)})$ in $M_n(\mathbb{C})$, the mixed determinant is defined as,*

$$D[\mathbf{A}] := \sum_{S_1 \cdots S_k = [n]} \det[A^{(1)}(S_1)] \cdots \det[A^{(k)}(S_k)].$$

Closely related to the mixed determinant is the following generalization of the characteristic polynomial of a single matrix to a tuple of k matrices, which we call the *mixed determinantal polynomial* (introduced in [Rav16])) or *MDP* in short.

Definition. *Given a k tuple of matrices $\mathbf{A} = (A^{(1)}, \cdots, A^{(k)})$ in $M_n(\mathbb{C})$, the mixed determinantal polynomial is defined as,*

$$\chi[\mathbf{A}](x) := k^{-n} D\left[xI - A^{(1)}, \cdots, xI - A^{(k)}\right].$$

This polynomial was shown to be real-rooted for hermitian arguments in [Rav16]. We show that the largest eigenvalues of k matrices are *simultaneously* controlled by the largest root of their MDP up to a factor of k, serving as a substitute for (I).

Theorem. *Let $A^{(1)}, \cdots, A^{(k)}$ be zero diagonal hermitian matrices. Then,*

$$\max_{i\in[k]} \lambda_{max}\, \chi(A^{(i)}) \leq k\, \lambda_{max}\, \chi[A^{(1)}, \ldots, A^{(k)}].$$

We will use the notation $A(S)$ to denote the submatrix of A indexed by rows and columns in S, and A_S to denote the matrix with rows and columns in S removed. For a collection of subsets $\mathbf{S} = S_1 \cup \ldots S_k$ we will use

$$A(\mathbf{S}) := \bigoplus_i A(S_i)$$

to denote the block matrix containing submatrices indexed by the S_i. Together with the *mixed determinantal polynomial* from above, we will also need a closely related polynomial,

Definition. *Let* $\boldsymbol{A} = (A^{(1)}, \cdots, A^{(k)})$ *be matrices in* $M_n(\mathbb{C})$ *and let* $S \subset [n]$*. Define the restricted mixed determinantal polynomial by,*

$$\chi[\boldsymbol{A}_S] := \chi[A_S^{(1)}, \cdots, A_S^{(k)}],$$

and analogously

$$\chi[\boldsymbol{A}(S)] := \chi[A^{(1)}(S), \cdots, A^{(k)}(S)]$$

Central to our main result will be characteristic polynomials and mixed determinantal polynomials of pavings.

Definition. *Let* $A \in M_n(\mathbb{C})$ *and let* $\boldsymbol{S} = \{S_1, \cdots, S_r\}$ *be a collection of subsets of* $[n]$*, that we will typically take to be a partition of* $[n]$*. Then, we define the characteristic polynomial of the collection,*

$$\chi[A(\boldsymbol{S})] := \prod_{i=1}^{r} \chi[A(S_i)].$$

Also, given matrices $\boldsymbol{A} = (A^{(1)}, \cdots, A^{(k)})$ *in* $M_n(\mathbb{C})$*, we define the mixed determinantal polynomial of the collection,*

$$\chi[\boldsymbol{A}(\boldsymbol{S})] := \prod_{i=1}^{r} \chi[\boldsymbol{A}(S_i)] = \prod_{i=1}^{r} \chi[A^{(1)}(S_i), \cdots, A^{(k)}(S_i)],$$

We then show that parts (II) and (III) of the interlacing families argument can be carried out for MDPs. As in previous works, (II) is established by deriving a differential formula for MDPs which shows that they and all of their relevant convex combinations are real-rooted, using the theory of real stability due to Borcea and Branden [BB10], [BB08].

Theorem. *Let* $\boldsymbol{A} = (A^{(1)}, \cdots, A^{(k)})$ *be a k tuple of hermitian matrices in* $M_n(\mathbb{C})$*. Then, there is a partition* $\boldsymbol{S} = (S_1, \cdots, S_r)$ *of* $[n]$ *such that,*

$$\lambda_{max}\left(\chi[\boldsymbol{A}(\boldsymbol{S})]\right) \leq \lambda_{max}\left(\mathbb{E}_{\boldsymbol{S} \in \mathscr{P}_r} \chi[\boldsymbol{A}(\boldsymbol{S})]\right) = \lambda_{max}\left(\chi[\overbrace{\boldsymbol{A}, \cdots, \boldsymbol{A}}^{r}]\right).$$

Finally, for step (III), the required bound follows by observing that the relevant expected characteristic polynomials are equivalent (after a change of variables) to the mixed characteristic polynomials of [MSS15], and appealing to the bounds derived there.

Theorem. *Let* $A^{(1)}, \cdots, A^{(k)}$ *be a k tuple of zero diagonal hermitian contractions, where* $k \geq 2$*. Then,*

$$\lambda_{max} \chi[A^{(1)}, \cdots, A^{(k)}] < \frac{3\sqrt{2}}{\sqrt{k}}.$$

Theorems (1), (1) and (1) together yield our announced result.

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Analogues of Dedekind Sums

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Abstract

In this talk, a transformation formula under modular substitutions is derived for a large class of generalized Eisenstein series. Appearing in the transformation formulae are generalizations of Dedekind sums involving the periodic Bernoulli function. Reciprocity theorems are proved for these Dedekind sums. Furthermore, as an application of the transformation formulae, relations between various infinite series and evaluations of several infinite series are deduced.

Introduction

The classical Dedekind sum $s(d,c)$, arising in the theory of Dedekind η-function, is defined by

$$s(d,c)=\sum_{n(modc)}\left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{dn}{c}\right)\right)$$

where the *sawtooth* function is defined by

$$((x))=\begin{cases} x-[x]-1/2, & \text{if } x\in\mathbb{R}\backslash\mathbb{Z};\\ 0, & \text{if } x\in\mathbb{Z},\end{cases}$$

with $[x]$ the floor function. One of the most important properties of Dedekind sums is the reciprocity formula

$$s(d,c)+s(c,d)=-\frac{1}{4}+\frac{1}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1}{dc}\right),$$

whenever c and d are coprime positive integers.

Dedekind sums first arose in the transformation formulas of $\log\eta(z)$, where $\eta(z)$ denotes the Dedekind eta-function defined by

$$\eta(z)=e^{\pi iz/12\infty}_{n=1}\left(1-e^{2\pi inz}\right),\quad \text{for } Im(z)>0.$$

There are several other functions such as Eisenstein series, which possess transformation formula similar to $\log\eta(z)$. Lewittes [6] has discovered a method of obtaining transformation formulas for certain generalized Eisenstein series

$$G(z,s,r_1,r_2)=\sum_{\substack{m,n=-\infty\\(m,n)\neq(-r_1,-r_2)}}^{\infty}\frac{1}{((m+r_1)z+n+r_2)^s}\quad \text{for } Re(s)>2 \text{ and } r_1,r_2\in\mathbb{R}.$$

In [1], Berndt gave a different account of the final part of Lewittes' proof. His new proof yielded elegant transformation formulas in which Dedekind sums or various generalizations of Dedekind sums appear. The results of [1] have been generalized in [3]. Berndt [6] derived a number of transformation formulas from the general theorem in [3]. Arising in the transformation formulae are various types of Dedekind sums, all of which satisfy reciprocity theorems. In [2], Berndt considered a more general class of Eisenstein series involving χ (non-principle primitive character modulo k)

$$G(z,s:\chi:r_1,r_2)=\sum_{\substack{m,n=-\infty\\(m,n)\neq(-r_1,-r_2)}}^{\infty}\frac{\chi(m)\overline{\chi}(n)}{((m+r_1)z+n+r_2)^s},$$

and developed transformation formulae for a wide class of functions involving characters including the natural character generalizations of $\log\eta(z)$. For $s=r_1=r_2=0$, in these formulas further generalizations of Dedekind sums as

$$s(d,c:\chi)=\sum_{n\ (modck)}\chi(n)\bar{B}_1(nd/c,\chi)\bar{B}_1(n/ck)$$

appear. Here $\bar{B}_n(x)$ denote the nth Bernoulli function defined by

$$\bar{B}_n(x)=\begin{cases}B_n(x-[x]), & \text{if } n\neq 1 \text{ or } x\notin\mathbb{Z},\\ 0, & \text{if } n=1 \text{ and } x\in\mathbb{Z},\end{cases}$$

with $[x]$ being the largest integer $\leq x$ and $B_n(x)$ is the nth Bernoulli polynomial defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi). \tag{1}$$

Also, $\bar{B}_n(x,\chi)$ are the generalized Bernoulli functions functions with period k, can be defined by [4, Theorem 3.1]

$$\bar{B}_n(x,\chi)=k^{n-1}\sum_{j=1}^{k-1}\overline{\chi}(j)\bar{B}_n\left(\frac{j+x}{k}\right), \tag{2}$$

for all real x and $n\geq 0$. These sums possess following reciprocity law

$$s(c,d:\chi)+s(d,c:\chi)=\bar{B}_1(\chi)\bar{B}_1(\overline{\chi}),$$

whenever $(c,d)=1$, $c,d>0$ and c or $d\equiv 0(modk)$. Meyer [11] concerned the following type Eisenstein series

$$G(z,s:\chi:r,h)=\sum_{\substack{m,n=-\infty\\(m,n)\neq(-r_1,-r_2)}}^{\infty}\frac{\chi(m)\overline{\chi}(n)e((mh_1+nh_2)/k)}{((m+r_1)z+n+r_2)^s},$$

for $r=(r_1,r_2)$, $h=(h_1,h_2)$, $Re(s)>2$ and $Im(z)>0$. He established transformation formulae and obtained various types of Dedekind sums for some special cases.

Also in [5] and [6], Berndt has used the transformation formulas to evaluate several classes of infinite series and established many relations between various infinite series.

In this talk, we present a transformation formula for a very large class of Eisenstein series defined by

$$G(z,s;A_\alpha,B_\beta;r_1,r_2)=\sum_{\substack{m,n=-\infty\\(m,n)\neq(-r_1,-r_2)}}^{\infty}\frac{f(\alpha m)f^*(\beta n)}{((m+r_1)z+n+r_2)^s},\ Re(s)>2,\ Im(z)>0$$

where $\{f(n)\}$ and $\{f^*(n)\}$, $-\infty<n<\infty$ are sequences of complex numbers with period $k>0$, and $A_\alpha=\{f(\alpha n)\}$ and $B_\beta=\{f^*(\beta n)\}$. Generalizations of Dedekind sums involving the periodic Bernoulli function appear in the transformation formulae. It is shown that these Dedekind sums obey reciprocity theorems. Moreover, transformation formulas contain many other interesting results as special cases. These results give the values of several interesting infinite series and yield relations between various infinite series.

Main Results

The results presented here are based on [1, 3].

We first show that the function $G(z,s;A_\alpha,B_\beta;r_1,r_2)$ can be analytically continued to the entire s–plane with the possible exception $s=1$.

[. *Let $z\in\mathbb{H}$, $Re(s)>2$ and $\beta\beta^{-1}\equiv 1(modk)$. Then,*

$$\begin{aligned}&\Gamma(s)G(z,s;A_\alpha,B_\beta;r_1,r_2)\\&=(-2\pi i/k)^s k\left(A\left(z,s;A_\alpha,\widehat{B}_{-\beta^{-1}};r_1,r_2\right)+e(s/2)A\left(z,s;A_{-\alpha},\widehat{B}_{\beta^{-1}};-r_1,-r_2\right)\right)\\&\quad+\lambda_{r_1}f(-\alpha r_1)\Gamma(s)\left(L(s;B_\beta;r_2)+e(s/2)L(s;B_{-\beta};-r_2)\right)\end{aligned}$$

where

$$A(z,s;A_\alpha,A_\beta;r_1,r_2)=\sum_{m>-r_1}f(\alpha m)\sum_{n=1}^{\infty}f(\beta n)e\left(n\frac{(m+r_1)z+r_2}{k}\right)n^{s-1}$$

and

$$L(s;A_\beta;\theta)=\sum_{n>-\theta}f(n\beta)(n+\theta)^{-s},\ \textit{for } Re\,(s)>1 \textit{ and } \theta \textit{ real.} \tag{3}$$

Note that since $L(s;B_\beta;r_2)$ can be analytically continued to the entire s–plane with the possible exception $s=1$ and since $A\left(z,s;A_\alpha,B_\beta;r_1,r_2\right)$ is entire function of s, $G(z,s;A_\alpha,B_\beta;r_1,r_2)$ can be analytically continued to the entire s–plane with the possible exception $s=1$.

Now, we present transformation formulas for the function $G(z,s;A_\alpha,B_\beta;r_1,r_2)$ whose proof is similar to [2, Theorem 2].

Theorem. *Define $R_1=ar_1+cr_2$ and $R_2=br_1+dr_2$, in which r_1 and r_2 are arbitrary real numbers. Let $\rho=\rho(R_1,R_2,c,d)=\{R_2\}c-\{R_1\}d$. Suppose first that $a\equiv d\equiv 0\,(modk)$. Then for $z\in\mathbb{K}$ and all s,*

$$\begin{aligned}&(cz+d)^{-s}\Gamma(s)G(Vz,s;A,B;r_1,r_2)\\&=\Gamma(s)G(z,s;B_{-b},A_{-c};R_1,R_2)-2i\Gamma(s)\sin(\pi s)L(s;A_c;-R_2)f^*(bR_1)\lambda_{R_1}\\&\quad+e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}f(c([R_2+d(j-\{R_1\})/c]-\nu))f^*(b(\mu c+j+[R_1]))\\&\qquad\times I(z,s,c,d,r_1,r_2)\end{aligned} \tag{4}$$

where $L(s;A_c;R_2)$ is given by (3) and

$$\begin{aligned}&I(z,s,c,d,r_1,r_2)\\&=\int_C u^{s-1}\frac{\exp(-((c\mu+j-\{R_1\})/ck)(cz+d)ku)}{\exp(-ku(cz+d))-1}\frac{\exp(((\nu+\{(dj+\rho)/c\})/k)ku)}{\exp(ku)-1}du.\end{aligned} \tag{5}$$

Here, we choose the branch of u^s with $0<\arg u<2\pi$. Also, C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u=0$ is the only zero of $(\exp(-ku(cz+d))-1)(\exp(ku)-1)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane.

Secondly, if $b\equiv c\equiv 0\,(modk)$, then for $z\in\mathbb{K}$ and all s,

$$\begin{aligned}&(cz+d)^{-s}\Gamma(s)G(Vz,s;A,B;r_1,r_2)\\&=\Gamma(s)G(z,s;A_d,B_a;R_1,R_2)-2i\Gamma(s)\sin(\pi s)L(s;B_{-a};-R_2)f(-dR_1)\lambda_{R_1}\\&\quad+e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1}f^*(-a([R_2+d(j-\{R_1\})/c]-\nu+d\mu))f(-d(j+[R_1]))I(z,s,c,d,r_1,r_2).\end{aligned} \tag{6}$$

We will also need the following theorem whose proof is similar to the proof of (4).

Theorem. *Under the conditions of Theorem 1, for* $a \equiv d \equiv 0\,(mod k)$ *we have*

$$\begin{aligned}
&(cz+d)^{-s}\Gamma(s)G(Vz,s;B_{-\beta},A_{-\alpha};r_1,r_2) \\
&= \Gamma(s)G(z,s;A_{\alpha b},B_{\beta c};R_1,R_2) - 2i\Gamma(s)\sin(\pi s)f(-\alpha bR_1)L(s,B_{-\beta c};-R_2) \\
&+ e(-s/2)\sum_{j=1}^{c}\sum_{\mu=0}^{k-1}\sum_{v=0}^{k-1} f(-\alpha b(\mu c+j+[R_1]))f^*(-\beta c([R_2+d(j-\{R_1\})/c]-v))I(z,s,c,d,r_1,r_2),
\end{aligned} \tag{7}$$

where $I(z,s,c,d,r_1,r_2)$ *is given by (5).*

We investigate these transformation formulas for two cases.

Case I: $s = r_1 = r_2 = 0$

Theorem 1 can be simplified when s is an integer. In this case, $I(z,s,c,d,r_1,r_2)$ can be evaluated by the residue theorem with the aid of (1). Therefore, if $s = -N$, where N is a non-negative integer, a simple calculation yields

$$\begin{aligned}
&I(z,-N,c,d,r_1,r_2) \\
&= 2\pi i k^N \sum_{m+n=N+2} B_m\left(\frac{c\mu+j-\{R_1\}}{ck}\right)B_n\left(\frac{v+\{(dj+\rho)/c\}}{k}\right)\frac{(-(cz+d))^{m-1}}{m!n!}.
\end{aligned} \tag{8}$$

Thus, the transformation formulas turns into:

Theorem. *Let* $z \in \mathbb{H}$. *If* $a \equiv d \equiv 0\,(mod k)$, *then*

$$\begin{aligned}
&\lim_{s\to 0}\Gamma(s)\left((cz+d)^{-s}G(Vz,s;A,B) - G(z,s;B_{-b},A_{-c})\right) \\
&= 2\pi i s\,(d,c;B_b,A_c) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - 2\pi i\frac{cz+d}{c}B_0(A)P_2\,(0,B_b)\,,
\end{aligned}$$

where

$$s\,(d,c;B_b,A_c) = \sum_{n=1}^{ck} f^*(bn)P_1\left(\frac{n}{ck}\right)P_1\left(\frac{dn}{c},A_c\right)$$

is called as the periodic Dedekind sum with $P_1\,(x) = x - [x] - 1/2$ *and*

$$P_r(x,A_c) = k^{r-1}\sum_{v=0}^{k-1} f(-cv)P_r\left(\frac{v+x}{k}\right).$$

If $b \equiv c \equiv 0\,(mod k)$, *then*

$$\begin{aligned}
&\lim_{s\to 0}\Gamma(s)\left((cz+d)^{-s}G(Vz,s;A,B) - G(z,s;A_d,B_a)\right) \\
&= 2\pi i s\,(d,c;A_{-d},B_{-a}) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - 2\pi i\frac{cz+d}{c}B_0(B)P_2(0,A_d).
\end{aligned}$$

The periodic Dedekind sum $s\,(d,c;B_b,A_c)$ satisfies the following reciprocity formula.

Theorem. *Let c and d be coprime positive integers with* $d \equiv 0\,(mod k)$. *For* $bc \equiv -1\ (mod d)$,

$$\begin{aligned}
&s\,(-c,d;A_c,B_{-b}) - s\,(d,c;B_b,A_c) \\
&= P_1\,(0,B_{-b})\,P_1\,(0,A_{-c}) - \frac{d}{c}B_0(A)P_2\,(0,B_b) - \frac{c}{d}B_0\,(B)\,P_2\,(0,A_c) - \frac{1}{2dc}B_0(B)B_2(A).
\end{aligned}$$

Case II: $s = 0$, and r_1 and r_2 arbitrary

In this case, transformation formulas given above are evaluated and more general periodic Dedekind sums appear:

Definition. *Let c and d be coprime integers with* $d \equiv 0\,(mod k)$ *and* $c > 0$. *For* $bc \equiv -1 \;(mod d)$, *the generalized periodic Dedekind sum* $s(d,c;A_b,A_c;x,y)$ *is defined by*

$$s(d,c;A_b,A_c;x,y) = \sum_{n=1}^{ck} f(bn) P_1\left(\frac{n+y}{ck}\right) P_1\left(\frac{d(n+y)}{c} + x, A_c\right).$$

Note that $s(d,c;A_b,A_c;0,0) = s(d,c;A_b,A_c)$. Moreover $s(d,c;A_b,A_c;x,y)$ is the natural generalization of the generalized Dedekind sum $s(d,c;x,y)$.

Also, a reciprocity formula for $s(d,c;B_b,A_c;R_2,R_1)$ can be derived.

Theorem. *Let c and d be coprime positive integers with* $d \equiv 0\,(mod k)$. *For* $bc \equiv -1 \;(mod d)$,

$$s(-c,d;A_c,B_{-b};-R_1,-R_2) - s(d,c;B_b,A_c;R_2,-R_1)$$
$$= P_1(R_2,B_{-b}) P_1(-R_1,A_{-c}) - \frac{1}{cd} B_0(B) P_2(cR_2 - dR_1, A) - \frac{d}{c} B_0(A) P_2(R_2,B_{-b}) - \frac{c}{d} B_0(B) P_2(R_1,A_c).$$

We conclude our talk with applications of transformation formulas. Taking some special values of $A = \{f(n)\}$ and $B = \{f^*(n)\}$ in the transformation formulas, we present evaluations of several infinite series such as

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n\left(e^{n\gamma/2} + e^{-n\gamma/2}\right)} + \sum_{n=1}^{\infty} \frac{\chi(n)}{n\left(e^{n\theta/2} + e^{-n\theta/2}\right)} = \frac{\pi}{8}$$

and

$$\sum_{n=1}^{\infty} (-1)^n \frac{\operatorname{csch}(n\pi)}{n^{4M+3}} = -(2\pi)^{4M+3} \sum_{m=0}^{2M+2} (-1)^m \frac{B_{2m}(1/2)}{(2m)!} \frac{B_{4M+4-2m}(1/2)}{(4M+4-2m)!},$$

which was first proved by Cauchy as cited by Berndt [6].

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On Biconservative Immersions into Semi-Riemannian Manifolds

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Abstract

In this talk, we would like to present a survey of our recent results on biconservative immersions into some semi-Riemannian manifolds.

Keywords. biconservative immersions, Biharmonic conjecture, semi-Riemannian space forms, product spaces

Introduction

Let (M^m, g) and (N^n, h) be some (semi-)Riemannian manifolds. Then, the energy functional E is defined by

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dv$$

for any smooth mapping $\psi : M \to N$, where $d\psi$ denotes the differential of ψ. A mapping ψ is said to be harmonic if it is a critical point of the energy functional E. It is well known that a harmonic mapping ψ satisfy the Euler-Lagrange equation

$$\tau_1(\psi) = 0,$$

where $\tau_1(\psi) = \mathrm{tr}\nabla d\psi$ denotes the tension field of ψ (See for example, [3]).

In 1964, Eells and Sampson proposed an infinite dimensional Morse theory on the manifold of smooth maps between Riemannian manifolds whereas their results describe harmonic maps more rigorously [4]. Further, J. Eells and L. Lemaire proposed the problem to consider the k-harmonic maps in [3]. A particular interest has the case $k = 2$. The bienergy functional is defined by

$$E_2(\psi) = \int_M |\tau_1(\psi)|^2 dv$$

for a smooth mapping $\psi : M \to N$. As a natural generalization of harmonic maps, we have the following definition: A map ψ is called *biharmonic* if it is a critical point of the bienergy functional E_2. In [7], G.Y. Jiang studied the first and second variation formulas of E_2 and he proved that the mapping ψ is biharmonic if the associated Euler-Lagrange equation

$$\tau_2(\psi) = 0, \tag{1}$$

is satisfied, where

$$\tau_2(\psi) = \Delta\tau(\psi) - tr\tilde{R}(d\psi, \tau(\psi))d\psi$$

is the bi-tension field and Δ is the rough Laplacian acting on the sections of $\psi^{-1}(TN)$.

Before we proceed, we would like to note that if the mapping $\psi : M \to N$ is a biharmonic isometric immersion, then M is called a biharmonic submanifold of N. By splitting $\tau_2(\psi)$ into its normal and tangential components, one may conclude Let $\psi : M^m \to N^n$ be an isometric immersion between two Riemannian manifolds. Then, ψ is biharmonic if and only if the equations

$$m\mathrm{grad}\,\|H\|^2 + 4\mathrm{tr}A_{\nabla^\perp H}(\cdot) + 4\mathrm{tr}(\tilde{R}(\cdot, H)\cdot)^T = 0 \tag{2}$$

and

$$\operatorname{tr}\alpha_{\psi}(A_H(\cdot),\cdot)-\Delta^{\perp}H+\operatorname{tr}(\tilde{R}(\cdot,H)\cdot)^{\perp}=0 \tag{3}$$

are satisfied, where A, H and α_ψ denote the shape operator, the mean curvature vector and second fundamental form of ψ, $\nabla^{\perp}$ is the normal connection of M and $\Delta^{\perp}$ is the Laplacian associated with $\nabla^{\perp}$.

Biconservative mappings

A mapping $\psi: M\to N$ satisfying the condition

$$\langle\tau_2(\psi),d\psi\rangle=0, \tag{1}$$

that is weaker than (1), is said to be biconservative. In particular, if $\psi=\mathrm{x}$ is an isometric immersion, then (1) is equivalent to

$$\tau_2(\mathrm{x})^T=0,$$

where $\tau_2(\mathrm{x})^T$ denotes the tangential part of $\tau_2(\mathrm{x})$. In this case, M is said to be a biconservative submanifold of N. By considering Proposition 1, one can conclude the following well-known proposition, (See for example [1]).

Let $\psi: M^m\to N^n$ be an isometric immersion between two Riemannian manifolds. Then, ψ is biconservative if and only if the equation (2) is satisfied.

Biconservative immersions into semi-Riemannian space-forms

Let $R^n_s(c)$ denote the semi-Riemannian space-form with dimension n, index s and constant sectional curvatures c, i.e.,

$$R^n_s(c)=\begin{cases}\mathbb{S}^n_s(c) & \text{if} \quad c>0\\ \mathbb{H}^n_s(c) & \text{if} \quad c<0\end{cases}$$

and $\psi: M\to R^n(c)$ be an isometric immersion. We will put $R^n_0(c)=R^n(c)$. By considering (2), we have ψ is biharmonic if and only if the fourth order partial differential equation

$$m\operatorname{grad}\|H\|^2+4\operatorname{tr}A_{\nabla^{\perp}_{\cdot}H}(\cdot)=0 \tag{1}$$

is satisfied, where m is the dimension of M.

The general properties of biconservative surfaces into Euclidean spaces, and in particular the properties of biconservative surfaces with constant mean curvature, i.e. CMC biconservative surfaces, were studied in [12]. In [11], all CMC biconservative surfaces in 4-dimensional space forms $R^4(c)$ were completely classified: For $c\neq 0$, they are PMC, i.e. their mean curvature vector field is parallel, and for $c=0$, they are either PMC or certain cylinders.

Most recently, in a joint work the author and Rüya Yeğin Şen studied biconservative surfaces with paralel normalized mean curvature vector in $\mathbb{E}^4$ and proved that such a surface is necessarily congruent to a rotational surface in $\mathbb{E}^4$, [13]. Namely, the following theorem has been obtained:

Theorem. *Let M be a PNMCV surface in the 4-dimensional Euclidean space $\mathbb{E}^4$ with a point $m\in M$ at which $f(m)>0, (\operatorname{grad}f)(m)\neq 0$, where f is the mean curvature of M. If M is biconservative, then there exists a neighborhood of m on which M is congruent to the simple rotational surface*

$$\mathrm{x}(s,t)=(\alpha_1(s)\cos t,\alpha_1(s)\sin t,\alpha_2(s),\alpha_3(s)) \tag{2}$$

with arc-length parametrized smooth profile curve $\alpha(s)=(\alpha_1(s),\alpha_2(s),\alpha_3(s))$,

$$\alpha_1(s)=\frac{1}{c_2 f(s)^{3/4}}$$

whose curvature and torsion are given by

$$\kappa(s) = f(s)\sqrt{1+c^2 f(s)}, \tag{3a}$$

$$\tau(s) = \frac{cf'(s)}{2\sqrt{f(s)}(1+c^2 f(s))}. \tag{3b}$$

Moreover, f satisfies

$$f(s)f''(s) - \frac{7}{4}f'(s)^2 + 4f(s)^4 + \frac{4}{3}c^2 f(s)^5 = 0.$$

On the other hand, in the particular case $n = m+1$, (1) turns into

$$S(\operatorname{grad}\|H\|) = c\operatorname{grad}\|H\|. \tag{4}$$

Biconservative hypersurfaces in semi-Riemannian space forms are studied in [2, 6, 10, 15, 16] and, most recently, in [17].

Biconservative immersions into product spaces

Consider an isometric immersion $\psi : M^m \to R^n(\varepsilon) \times \mathbb{R}$, where $\varepsilon = \pm 1$. Let ∂_t be a unit vector field tangent to the second factor. Then, a tangent vector field T on M^m and a normal vector field η along f are defined by

$$\partial_t = f_* T + \eta. \tag{5}$$

We have the following definition given in [14] and extended to submanifolds of $R^n(\varepsilon) \times \mathbb{R}$ in [9]. We will denote by $\mathscr{A}$ the class of isometric immersions $\psi : M^m \to R^n(\varepsilon) \times \mathbb{R}$ with the property that T is an eigenvector of all shape operators of ψ.

On the other hand, from the expression of the curvature tensor of $R^n(\varepsilon) \times \mathbb{R}$ that f is biconservative if and only if

$$\langle H, \eta \rangle T = 0, \tag{6}$$

where η and T denote the vector fields giving in (5).

In [5], biconservative surfaces with parallel mean curvature vector field in $\mathbb{S}^n \times I\!R$ and $\mathbb{H}^n \times I\!R$ is classified. Recently, in [8] this study is extended into submanifolds of higher dimension. In [8], the author, Manfio and Upadhyay studied some of geometrical properties of biconservative submanifolds in $R^n(\varepsilon) \times \mathbb{R}$. Moreover, a classification of biconservative submanifolds in $R^4(\varepsilon) \times \mathbb{R}$ is given. More precisely, it is proved that a biconservative submanifold in $R^4(\varepsilon) \times \mathbb{R}$ with nonzero parallel mean curvature vector field and codimension 2 belongs to class $\mathscr{A}$.

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Binary three-weight linear codes from partial geometric difference sets

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Abstract

Links between linear codes, non-linear functions from cryptography, graphs and combinatorial designs have attracted the attention of many researchers over the last 50 years. Difference set method is a powerful tool to construct designs and explore the links between designs and many other combinatorial objects including codes and nonlinear cryptographic functions.

In this talk, we will introduce a generalisation of $(v,k,\lambda)-$difference sets known as partial geometric difference sets. In particular, we will show that existence of a family of partial geometric difference sets is equivalent to existence of a certain family of three-weight linear codes. We also provide a link between binary plateaued functions, three-weight linear codes and partial geometric difference sets.

An Announcement of new Results on the Periodicity of the Jacobi-Perron Algorithm

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Abstract

We present some of the results obtained in [6], including evidence that the answer to the long-standing question about the periodicity of the Jacobi-Perron Algorithm is no.

We propose that we have found all integers $1 \leq m \leq 10^8$ such that the Jacobi-Perron algorithm for $\left(m^{1/3}, m^{2/3}\right)$ is periodic.

In fact, we conjecture that the values of m with such periodic JPA's have density 0 in the set of positive integers,

Introduction

The Jacobi-Perron Algorithm (JPA) is one of the generalisations of the ordinary continued fraction algorithm to higher dimensions. It has been the subject of considerable study (see, for example, [1,5]).

Let $\alpha^{(0)} = \left(\alpha_1^{(0)}, \ldots, \alpha_{n-1}^{(0)}\right) \in \mathbb{R}^{n-1}$ for $n \geq 2$. The JPA expansion of $\alpha^{(0)}$ is the sequence of elements of $\mathbb{R}^{n-1}$, $\langle \alpha^{(\nu)} \rangle_{\nu \geq 0}$, defined by

$$\begin{aligned}
\mathbf{a}^{(\nu)} &= \left(a_1^{(\nu)}, \ldots, a_{n-1}^{(\nu)}\right) \\
&= \left(\lfloor \alpha_1^{(\nu)} \rfloor, \ldots, \lfloor \alpha_{n-1}^{(\nu)} \rfloor\right), \\
\alpha^{(\nu+1)} &= \left(\alpha_1^{(\nu+1)}, \ldots, \alpha_{n-2}^{(\nu+1)}, \alpha_{n-1}^{(\nu+1)}\right) \\
&= \left(\frac{\alpha_2^{(\nu)} - a_2^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \ldots, \frac{\alpha_{n-1}^{(\nu)} - a_{n-1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}}\right),
\end{aligned}$$

where $\alpha_1^{(\nu)} \neq a_1^{(\nu)}$ and $\lfloor \cdot \rfloor$ is the greatest integer function.

For $n = 2$, this is the ordinary continued fraction algorithm. When $n = 3$, this algorithm was given by Jacobi [2] and is sometimes called the *Jacobi Algorithm*. Perron [4] extended Jacobi's definition to arbitrary n. As a result, for all $n \geq 2$, the algorithm is now known as the *Jacobi-Perron Algorithm*. We have presented it here in its inhomogeneous form.

The JPA expansion of $\alpha^{(0)}$ is called *periodic*, if there exist two integers ℓ_0, ℓ_1 with $\ell_0 \geq 0$ and $\ell_1 \geq 1$ such that

$$\alpha^{(\nu+\ell_1)} = \alpha^{(\nu)} \text{ for } \nu = \ell_0, \ell_0 + 1, \ldots.$$

If ℓ_0 and ℓ_1 are the smallest integers satisfying these conditions, then

$$\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\ell_0-1)} \text{ and } \alpha^{(\ell_0)}, \alpha^{(\ell_0+1)}, \ldots, \alpha^{(\ell_0+\ell_1-1)}$$

are called respectively the *preperiod* and the *period* of the periodic JPA expansion, and ℓ_0 and ℓ_1 are their respective *lengths*. If $\ell_0 = 0$, then the JPA expansion of $\alpha^{(0)}$ is said to be *purely periodic*.

Lagrange showed that the ordinary continued fraction expansion of a real number α is periodic if and only if α is a quadratic irrational. A central problem in the study of multidimensional generalisations of the ordinary continued fraction expansion is to find, and prove, an analogue of this result. The following question is part of JPA folklore.

When $1, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ form a $\mathbb{Q}$-basis of a number field of degree n, is the JPA expansion of $(\alpha_1, \ldots, \alpha_{n-1})$ periodic?

In this talk, we present numerical evidence suggesting that this is not always true for $n \geq 3$. In particular, our computation work suggests the following.

The set of all cubefree integers, m, such that the JPA expansion of $\left(m^{1/3}, m^{2/3}\right)$ is periodic has density 0 in the integers.

Computation Details

Two large-scale computations were done.

First, for all integers, $1 \leq m \leq 10^9$, that are not perfect cubes, we calculated the first 200 terms of the Jacobi-Perron expansion of $\left(m^{1/3}, m^{2/3}\right)$.

This was done using the PARI computer algebra system [3]. The general number field structures and functions such as nfeltadd, nfbasistoalg, ... were used, as these were faster than the naive implementation of the algorithm using floating point arithmetic. Because of the scale of the computations, the scripts were first written in PARI's GP scripting language and then compiled to C using the gp2c compiler provided by PARI. This, as well as the execution of the executable file produced in this manner, was done within an Ubuntu 15.04 virtual machine running in the Oracle VM VirtualBox Manager on a Windows 10 laptop with an Intel Core i7-3630QM CPU running at 2.40GHz. This processor has four physical cores, all of which were simultaneously used for this computation. The cumulative time taken to complete this computation was just over 7000 hours.

Motivated by some observations of the results of this first computation, a second computation was undertaken.

The second computation searched for longer periods, but over a smaller range of values of m. For all integers, $1 \leq m \leq 10^8$, that are not perfect cubes, we calculated the first 1000 terms of the Jacobi-Perron expansion of $\left(m^{1/3}, m^{2/3}\right)$. The code used, as well as the environment (both software and hardware), was the same as for the first computation. The cumulative time taken to complete this computation was approximately 5000 hours.

Computation Results

We summarise here the results obtained in the following two tables and our reasons for the conclusions that we draw from these computations.

It was as a result of observing the records for the sum of the preperiod and period in the results of the first computation that the idea to attempt to find all periodic JPAs via the second computation was undertaken.

The three last records obtained from the first calculation were:
$m = 49844864$, period=171, preperiod= 8, sum=179,
$m = 79510225$, period=139, preperiod= 55, sum=194,
$m = 535367840$, period=165, preperiod= 34, sum=199.

This led us to wonder what larger records might be observed if we searched over a larger number of terms in the JPA expansions. Doing so, as described in the previous section, led to the records in Table 1, which we believe are the actual records for all periodic JPA expansions in these ranges.

Together both tables suggest that all periodic examples with $m \leq 10^8$ have been found. From Table 2, we see that very few (7, in fact) new examples were found by extending the search from the first 200 terms in the JPA expansion to the first 1000 terms of these expansions.

sum	period	preperiod	m
93	61	32	17
104	55	49	5346
104	35	69	21840
104	94	10	52731
132	41	91	75547
174	170	4	779792
214	175	39	9533216
224	174	50	13138592
241	105	136	42877625

Table 1: Period Size Records

m	periodic count (1000)	periodic count (200)
10^2	33	33
10^3	99	99
10^4	297	297
10^5	806	806
10^6	2129	2129
10^7	5385	5383
10^8	13418	13411
10^9		32531

Table 2: Periodic Counts

Furthermore, the record sizes of the sum of the preperiod and period are small, compared to 1000, with their growth slow and relatively regular. The extensive data produced and the apparent behaviour observed in the data give very little reason to believe that there are periodic examples in this range for which the sum of the period and preperiod more than quadruple the observed records as well as no examples at all where this sum is between 241 and 1000. In fact, we believe that the data gives every reason to believe no such examples exist and so our claim about finding all such examples is true.

Lastly, the above observations and conjectures about the completeness of this data, the counts themselves and also the growth rate of the counts support our conjecture that the density of such periodic expansions is 0.

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On the cubic polynomial Hamiltonian differential systems and the limit cycles bifurcating from their periodic solutions

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center, limit cycle, averaging method, phase portrait, cubic Hamiltonian systems.

Abstract

In this paper first we classify the phase portraits in the Poincaré disc of a class of Hamiltonian cubic polynomial differential systems having five centers. Moreover using the averaging theory up to sixth order, we study the number of limit cycles wich can bifurcate from the centers of the previous differential system when we perturb them inside the class of all polynomial differential systems of degree 3.

Introduction and statement of the main results

K.C. Sibirskii [9] in 1954 proved that at most five limit cycles can appear by a Hopf bifurcation in a cubic polynomial differential system without quadratic terms. J. Li and Q. Huang [6] in 1987 constructed a cubic polynomial differential system with 11 limit cycle. In 1995 H. Zoladek [11] proved that surrounding a focus of a cubic polynomial differential system there may exist 11 limit cycles. In 2005 P. Yu and M. Han [10] gave an example of a cubic polynomial with 12 limit cycles in a $(6,6)$ distribution. In 2009 C. Li, C. Liu and J. Yang [5] constructed a planar cubic system with at least 13 limit cycles. In 2014 Y.R. Liu and J. Li [8] show that there are planar Z_2–equivariant cubic polynomial differential systems having two elementary foci with at least 13 limit cycles. The main objective of this paper is to the number of limit cycles wich can bifurcate from the centers of a new classes of cubic polynomial differential systems.

First we study the phase portraits of the cubic polynomial differential systems

$$\begin{aligned} \dot{x} &= -y(1+by)(1+2by), \\ \dot{y} &= x(1+ax)(1+2ax). \end{aligned} \tag{1}$$

We note that this differential system is a Hamiltonian differential system with the Hamiltonian

$$H = \frac{1}{2}\left((x+ax^2)^2 + (y+by^2)^2\right). \tag{2}$$

Our first result is the following.

Theorem. *A polynomial differential system* (1) *with* $a^2+b^2 \neq 0$ *has a phase portrait in the Poincaré disc topologically equivalent to one of the two phase portraits of Figures 1 and 2.*

Figure 1: Case $a > 0$ and $b > 0$. The separatrices of this phase portrait are the circle at infinity, the four saddles and the five centers described in statement (a) of Proposition **??**, and the eight heteroclinic orbits formed by the separatrices of the four saddles. Therefore this phase portrait has six canonical regions.

Figure 2: Case $a = 0$ and $b > 0$. The separatrices of this phase portrait are the circle at infinity, the saddle and the two centers described in statement (b) of Proposition **??**, and the two homoclinic orbits formed by the separatrices of the saddle. Therefore this phase portrait has three canonical regions.

For the definition of Poincaré disc and the local charts for studying it see, for instance, Chapter 5 of [3]. The definitions of topological equivalent phase portraits, separatrices and canonical regions can be found in Chapter 1 of [3].

We consider the following perturbed cubic polynomial differential systems of system (1)

$$\begin{aligned} \dot{x} &= -y(1+by)(1+2by) + \sum_{s=1}^{6} \varepsilon^s p_i(x,y), \\ \dot{y} &= x(1+ax)(1+2ax) + \sum_{s=1}^{6} \varepsilon^s q_i(x,y), \end{aligned} \tag{3}$$

where

$$\begin{aligned} p_j(x,y) = &\quad \alpha_1^j y + \alpha_2^j x^2 y + \alpha_3^j y^3, \\ q_j(x,y) = &\quad \beta_1^j x + \beta_2^j x y^2 + \beta_3^j x^3, \end{aligned}$$

being α_i^j and β_i^j, for $i = 1,2,3$ and $j = 1,\ldots,6$, real constants.

Theorem. *For $\varepsilon > 0$ sufficiently small there are cubic polynomial differential systems* (3) *with at least* 10 *limit cycles.*

We shall study how many limit cycles can bifurcate from the five centers of the differential system (1) when a and b are positive using the averaging theory up to order 6. From the proof of Theorem 1 we shall see that at least 2 limit cycles can bifurcate from the centers $(-1/a, 0)$ and $(0, -1/b)$. We shall obtain similar results for the other three centers $(0,0)$, $(-1/a, -1/b)$ and $(-1/(2a), -1/(2b))$.

The second objectif of ower work is to study how many limit cycles can bifurcate from the five centers of the differential system (1) when a and b are positive using the averaging theory up to order 6.

We consider the following cubic perturbation of the cubic polynomial differential system (1)

$$\begin{aligned} \dot{x} &= -y(1+by)(1+2by) + \sum_{s=1}^{6} \varepsilon^i p_i(x,y), \\ \dot{y} &= x(1+ax)(1+2ax) + \sum_{s=1}^{6} \varepsilon^i q_i(x,y), \end{aligned} \tag{4}$$

where

$$\begin{aligned} p_j(x,y) = &\quad \alpha_0^j + \alpha_1^j x + \alpha_2^j x^3 + \alpha_3^j x^2 y + \alpha_4^j x y^2 + \alpha_5^j y^3, \\ q_j(x,y) = &\quad \beta_0^j + \beta_1^j y + \beta_2^j x^3 + \beta_3^j x^2 y + \beta_4^j x y^2 + \beta_5^j y^3, \end{aligned}$$

being α_i^j and β_i^j, for $i = 0,\ldots,5$ and $j = 1,\ldots,6$, real constants.

Theorem. *For $\varepsilon > 0$ sufficiently small the number of limit cycles of the differential system* (4) *with $a > 0$ and $b > 0$ bifurcating from the center* $(0,0)$ *obtained using the averaging theory of order*

(a) one and two is 0*;*

(b) three and four is 1*;*

(c) five and six is 2*;*

Similar results to the ones obtained in Theorem 1 for the center at the origin of system (4) with $a > 0$ and $b > 0$ can be obtained for the centers $(-1/a, -1/b)$ and $(-1/(2a), -1/(2b))$.

Finally we shall study how many limit cycles can bifurcate from the each one of the two centers of the differential system (1) when $a = 0$ and $b > 0$.

Theorem. *For $\varepsilon > 0$ sufficiently small the number of limit cycles of the differential system* (4) *with $a = 0$ and $b > 0$ bifurcating from the center, either* $(0,0)$*, or* $(0, -1/b)$*, obtained using the averaging theory of order*

(a) one and two is 0*;*

(b) three and four is 1*;*

(c) five and six is 2*;*

.

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Manifolds with Corners in Diffeology

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Abstract

We embed the category of manifolds with corners into the category {Diffeology} by the process of modeling, and we prove a theorem of extension for differential forms on corners and on manifolds with boundary.

Introduction

Manifolds with corners have been introduced long ago in the usual framework of differential geometry, for example as *variétés à bord généralisées* by Cerf [3, Chap. 1 §1.2], and then as *variétés à bords anguleux* by Douady [5, §4]. Over time the various descriptions of manifolds with boundary or corners evolved to a commonly accepted definition, based on the heuristic that a real smooth map defined on a corner should be defined as the restriction of a smooth map defined on an open neighborhood of the corner.

By the general procedure of *modeling diffeologies* [10], one can define what we understand as *manifolds with corners* in the category {Diffeology}. They are natural models for the unified procedure of modeling spaces in diffeology. Half-spaces, $H^n = [0,\infty[\times\mathbb{R}^{n-1}$, are the models for the category of *manifolds with boundary*. Corners, $K^n = [0,\infty[^n$, are the models for the category of *manifold with corners*.

Precisely,

Definition. *A n-*manifold with corners *is a diffeological space diffeomorphic to the corner K^n at each point.*

A natural question is then to compare that definition with the traditional approach introduced originally in [3, 5], and then used or developed by many authors, for example [**?**, 6, 7, 12, etc.]. In this approach, smooth maps from corners into the real line are — by definition — the restrictions of smooth maps on some open neighborhood of the corner [3] [5] etc. In Diffeology, this heuristic becomes a theorem, where the corners are equipped with the subset diffeology. Precisely:

Theorem. *Every map from K^n to $\mathbb{R}$ such that composed with a smooth parametrisation[1] $P\colon U \to \mathbb{R}^n$, taking its values in K^n, is smooth, is the restriction of a smooth maps defined on some open neighborhood of the corner.*

Thanks to this theorem, we show that the two aproaches define the same objects, and then the same category. Hence, as a subcategory of {Diffeology}, manifolds with corners inherit automatically all the diffeological constructions: smooth maps, fiber bundles, homotopy, differential calculus, homology, cohomology, etc.

It is always a progress when a convention, based on mathematicians' intuition, becomes a theorem in a well defined axiomatic. Here the axiomatic is the theory of Diffeology.

Now, noticing that $\mathscr{C}^\infty(K^n,\mathbb{R})$ is the space of differential 0-forms $\Omega^0(K^n)$, it is legitimate to ask about the behavior of differential k-forms on K^n, that is, $\Omega^k(K^n)$ as it is defined in [10]. Next, it has already been proved that differential forms on a half-space can be extended on a neighborhood of the half-space [8]. We show that this property is also satisfied by manifolds with boundary. Precisely,

[1] A parametrisation is just a map defined on an open subset of an Euclidean space.

Theorem. *Let M be a manifold with boundary imbedded in some manifold N as a* pièce à bord *[5]. If ∂M is compact, then every differential k-form on M extends on a neighborhood of M in N.*

For the purpose of this question, we had to establish a diffeological version of Taylor's series for real functions depending smoothly on parameters running in a diffeological space, and, a version of Whitney's theorem on extension of smooth real even functions.

Considering corners, we prove the extension theorem:

Theorem. *Every differential form on the corner K^n is the restriction of a smooth form on an open neighborhood of K^n in $\mathbb{R}^n$. Precisely, the pullback : $j^* : \Omega^k(\mathbb{R}^n) \to \Omega^k(K^n)$ is surjective, where j denotes the inclusion from K^n into $\mathbb{R}^n$.*

Acknowledgments

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Ehrhart Interpolation Polytopes

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Abstract

In order to compute the Ehrhart polynomial of a lattice polytope, we are constructing a family of polytopes called Ehrhart Interpolation Polytopes. This construction is based on structural properties of Ehrhart polynomials. The study of Ehrhart Interpolation Polytopes for specific properties can help in the computation of Ehrhart polynomials. On the other hand, it may also lead in a better understanding of which vectors are candidates for being h^*vectors of polytopes.

Introduction

Let $\mathscr{P} \subseteq^d$ a full dimensional polytope and L a lattice in ^{n}. For any positive integer t, let $tP = \{tx : x \in \mathscr{P}\}$ be the t-fold dilation of $\mathscr{P}$ and $\mathscr{L}_{\mathscr{P}}(t) = \#(t\mathscr{P} \cap L)$ be the number of lattice points contained in $t\mathscr{P}$.

Let $\mathscr{P}$ be a lattice polytope, i.e., the vertices of $\mathscr{P}$ are lattice points. It is known, due to Ehrhart [1], that there exist rational numbers $a_0, \ldots, a_d$ such that: $\mathscr{L}_{\mathscr{P}}(t) = a_d t_d + a_{d1} t_{d1} + \cdots + 1$ for all $t \in^*$. We call $\mathscr{L}_{\mathscr{P}}(t)$ the Ehrhart polynomial of $\mathscr{P}$.

The original motivation for the work presented here is to compute the Ehrhart polynomial. The problem is computationally hard. Nevertheless, since it is a polynomial, we would like to use some form of interpolation. The straighforward way to intepolate is to compute $d+1$ values. But since computing the number of lattice points in a polytope, which is required for computing values of $\mathscr{L}_{\mathscr{P}}(t)$, is also computationally heavy, we wish to reduce the number of required values, by taking advantage of other properties of Ehrhart polynomials.

Although assuming knowledge for less than $d+1$ values of thee Ehrhart polynomial is a first step, we would prefer to lift this restriction as well. The number of lattice points in a polytope is called the discrete volume, which is an appropriate term, since for *large enough* polytopes it is close to the (normalized) Euclidean volume. That is, the Ehrhart polynomial approximates Euclidean volume as t goes to infinity.

Efficient approximation of Euclidean volume is possible, and there are efficient implementations as well (see [2]). We will use volume approximation in order to get bounds for the number of lattice points in *large enough* dilations of our polytope.

Given a polytope $\mathscr{P}$, we will denote by $\mathbf{m}t$ and M_t, a lower and an upper bound respectively, for the number of lattice points in $t\mathscr{P}$.

Ehrhart Interpolation Polytopes

In our effort to interpolate Ehrhart polynomials, we use structural information about them. In this section, we define the notion of Ehrhart Interpolation Polytope. The main idea is that we can write Ehrhart polynomials in the binomial basis of the polynomial ring, and then the coefficients are non-negative integers. This means that computing the Ehrhart polynomial is equivalent to finding the corresponding coefficient vector (called the h^*vector) among the lattice points in the positive orthant. Except for the non-negativity condition, there are more inequalities that h^*vectors have to satisfy. If we take these into account as well, we obtain a polytope, which we call the Ehrhart Interpolation Polytope (see Definition 1).

Let d be the dimension of the polytope $\mathscr{P}$ and consequently the degree of $\mathscr{L}_{\mathscr{P}}(t)$. It is known (see [3]) that the generating function of the Ehrhart polynomial, called the Ehrhart series, can be written as

$$\sum_{t\geq 0} \mathscr{L}_{\mathscr{P}}(t) z^t = \frac{\sum_{j=0}^{d} h_j^* z^j}{(1-x)^{d+1}}.$$

From [3] again, we have that

$$\mathscr{L}_{\mathscr{P}}(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d} \text{ and } h_j^* \in \tag{1}$$

The vector $\left(h_0^*, h_1^*, \ldots, h_d^*\right)$ is called the h^*vector of $\mathscr{P}$. We fix the binomial basis as the basis of the polynomial ring for the rest of this abstract.

Given a polytope $\mathscr{P} \subseteq^d$, the h^*vector is a vector in $^{d+1}$. The h^*vector was the subject of many studies in the last decades. A lot of interesting results exist, but we will only discuss the ones relevant for the definition of the Ehrhart Interpolation Polytope, namely inequalities that the h^*vector of a polytope has to satisfy. Let us denote the set of such inqualities (I) in this abstract.

In [4], Stanley proves the following monotonicity theorem.

Theorem ([4])**.** *If $\mathscr{P}$ and $\mathscr{Q}$ are polytopes in ^{n} and $\mathscr{P} \subseteq \mathscr{Q}$, then $h^*_{\mathscr{P},i} \leq h^*_{\mathscr{Q},i}$, where $h^*_{\mathscr{P}}$ and $h^*_{\mathscr{Q}}$ are the h^*vectors of $\mathscr{P}$ and $\mathscr{Q}$ respectively.*

In order to devise upper bounds for the h^*vector of a given polytope $\mathscr{P}$, we construct the smallest cube C containing $\mathscr{P}$. The h^*vectors of hypercubes are easy to compute and this way we bound from above all coordinates of the h^*vector of $\mathscr{P}$ as follows

$$h^*_{\mathscr{P},i} \leq h^*_{C,i} \text{ for } 0 \leq i \leq d. \tag{2}$$

We add these inequalities to the set (I) as well.

If we assume we know the number of lattice points in the t-th dilation of $\mathscr{P}$, for some $t \in^*$, then we can add the following equation based on Eq 1

$$\sum_{i=0}^{d} h_i^* \binom{t+d-i}{d} = \mathscr{L}_{\mathscr{P}}(t) \tag{3}$$

to our list of constraints (I). Note that we only assume knowledge of $\mathscr{L}_{\mathscr{P}}(t)$ for one value t.

Now, let us define the Ehrhart Interpolation Polytope.

Definition (Ehrhart Interpolation Polytope)**.** *Given a polytope $\mathscr{P} \subseteq^d$ and $t \in^*$, we define*

$$E(t) = \left\{ x \in^{d+1} : x \text{ satisfies } (I) \right\} \subseteq^{d+1} \tag{4}$$

to be the Ehrhart Interpolation Polytopeof $\mathscr{P}$ in dilation t.

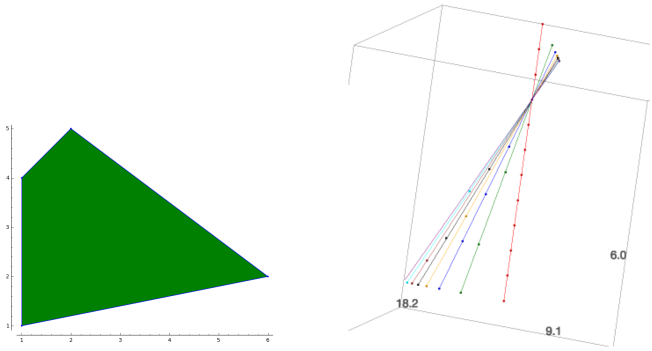

Figure 3: The Ehrhart Interpolation Polytopeis on the right. Notice that the purple segment contains only a single lattice point, the h^*vector of the polytope in the left.

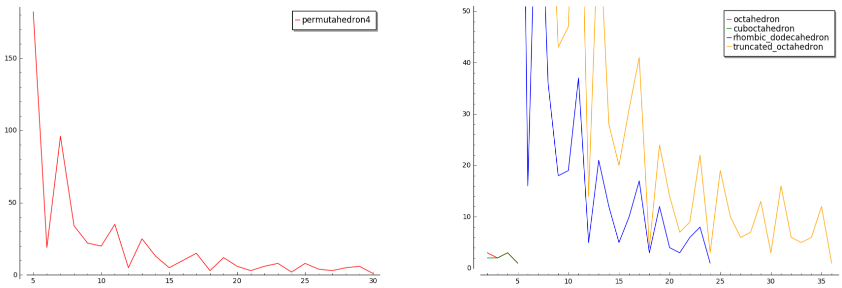

Figure 4: Number of lattice points in the Ehrhart Interpolation Polytopefor the permutahedron in dimension 4 (left) and for some solids (right) for dilations 5 to 30.

Note that the Ehrhart Interpolation Polytopeis indeed a bounded polyhedron, since it is contained in the intersection of the positive orthant with a hyperplane whose normal is a strictly positive vector (containing binomials).

[. *Let* $\mathscr{P} \subseteq^2$ *be the convex hull of the points* $(1,1),(1,4),(2,5),(6,2)$*. Then the Ehrhart Interpolation Polytopeas defined above is an* 1*-dimensional polytope in* ^{3}*. In figure 1 we can see the original polytope* $\mathscr{P}$ *and the Ehrhart Interpolation Polytopefor 8 values of* t*, namely* $t = 1,2,\ldots,8$.

For $t = 1,2,\ldots,8$*, there are* $12,3,6,2,3,2,3,1$ *lattice points in each segment respectively. For* $t = 8$ *(purple), the line segment contains only one lattice point. This lattice point is the* h^**vector of the polytope* $\mathscr{P}$*, contained naturally in the Ehrhart Interpolation Polytopefor all dilations.*

We explore properties of the Ehrhart Interpolation Polytope, such as the dilation for which there is only one lattice point contained in it. This implies that the single lattice point is the h^*vector of the original polytope. See Figure 1 for examples.

Finally, our ultimate goal is to use volume approximation instead of knowledge about the exact number of lattice points in a dilation. The main difference is using the bounds $\mathbf{m}t$ and M_t instead of the exact count, which means that instead of having a hyperplane, we have a thicker slice as part of the Ehrhart Interpolation Polytopedefinition.

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Interpretable Fields in ACF

Zeynep Kısakürek

Abstract

One of the main interests of model theory is to specify the definable sets, that are given by a formula, in a given structure. In this aspect, the theory of algebraically closed fields *ACF* is quite rich. It is well-known that, by quantifier elimination, definable sets in algebraically closed fields are exactly the constructible sets in algebraic geometric sense. This is not the only interaction between model theory and algebraic geometry. In this work, we focus on some results regarding the aforementioned interaction. We present that the characterization of definable fields in an algebraically closed field which is provided by Bruno Poizat [**?**] by introducing some model theoretic notions and results about linear algebraic groups [**?**], [**?**].

Keywords: Model Theory of Algebraically Closed Fields, Groups of Finite Morley Rank

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A Survey On Opial Inequalities

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Abstract

In this talk, we give a short review on Opial type inequalities.

Integral inequalities have been frequently employed in the theory of functional analysis, differential equations and applied sciences such as probablity and statistics. There are a lot of types of inequalities such as Hermite-Hadamard type inequalities, Lyapunov type inequalities, Wirtinger type inequalities and Qi type inequalities.They have been the center of attention in many papers. Especially, Opial inequalities have been studied by many authors.

In the year 1960, Opial [6] established the following fascinating integral inequality which is called Opial inequality :

Theorem. *(Opial Inequality)Suppose* $f(t) \in C^{(1)}[0,h]$ *is such that* $f(0) = f(h) = 0$*, and* $f(t) > 0$ *in* $(0,h)$*. Therefore, the following inequality is valid*

$$\int_0^h \left| f(t)f'(t) \right| dt \leq \frac{h}{4} \int_0^h (f'(t))^2 dt. \tag{5}$$

Moreover, in (5), the constant $h/4$ *is the best possible.*

Recently, various form and improvements of Opial inequalities have been given in the literature. Over the last three decades, generalizations in various directions have been investigated, and Opial type inequalities have become a subject in its own right. Opial inequality has been stimulating much interest of many mathematicans and Opial type inequalities have been established on time scale. We present some theorems on generalizations of Opial inequality and Opial type inequalities on time scale. We survey the literature on those type inequalities. **2010 MSC Codes.** Primary 26D15; Secondary 26D10.
Keywords: Opial Inequalities, Inequalities, Time Scales.

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Classification of Lattès Maps

Melike Efe

Abstract

The purpose of this work is to investigate Lattès maps on $\widehat{\mathbb{C}}$ which are holomorphically conjugate to an affine map on $\mathbb{C}/\Lambda$. In this work, we introduce some notions and facts from dynamical systems, algebraic topology and complex analysis in order to examine these maps deeply. A part of our work concerns the results of John Milnor related to Lattès maps. Specifically, we will see that the degree of a conjugating holomorphism is either $2, 3, 4$ or 6. Following this, we introduce the explicit form of a conjugating holomorphism of a given degree by using the aforementioned results in addition with the property that an elliptic function can be written as a rational function of Weierstrass' elliptic function and its derivative. Finally, we describe the ramification behaviour of Lattès maps.

Keywords: Lattès maps, ramification behaviour, Weierstrass' elliptic function, Riemann-Hurwitz formula, dynamical systems.

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Plancherel Formula for SU(2)

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Abstract

In this work, we study a constructive proof of the Plancherel formula for the compact group $SU(2)$ by using the Weyl's integration formula. The arguments of the proof make it possible to generalize the formula to non-compact groups $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$. The work consists of several parts. First, we derive the Plancherel formula for finite groups. In the second part, we generalize it to arbitrary compact groups by proving the Peter-Weyl Theorem. In final part, we classify the irreducible representations of $SU(2)$ and give a constructive proof as stated at beginning.

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