A Framework for a Valuative Nagata Conjecture for Surfaces

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The (classical) Nagata conjecture is one of the most stimulating problems for linear systems on the complex projective plane $\mathbb{P}^2$. It predicts the inequality $d > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i$, where d is the degree of any curve C on $\mathbb{P}^2$ such that $\text{mult}_{x_i}(C) \geq m_i$, m_i being non-negative integers and $\{x_i\}_{i=1}^{r}$, $r \geq 10$, very general points in $\mathbb{P}^2$. Nagata proved this result when r is a square and it is an open problem in the remaining cases.

The Seshadri-type constant $\hat{\mu}_D(\nu)$ of a divisorial valuation ν of a surface S relative to a big divisor D on S is defined to be

$$\hat{\mu}_D(\nu) := \lim_{m\to\infty} \frac{\max\{\nu(f) \mid f \in H^0(S, \mathcal{O}_S(mD))\}}{m}.$$

The valuation ν is called minimal with respect to a big divisor D on S if

$$\hat{\mu}_D(\nu) = \sqrt{\frac{\text{vol}_S(D)}{\text{vol}(\nu)}}.$$

The **valuative Nagata conjecture** states that if ν is a very general plane valuation **of** $\mathbb{P}^2$ such that its normalized volume satisfies $[\text{vol}^N(\nu)]^{-1} \geq 9$, then ν is minimal.

This conjecture implies the classical Nagata conjecture and it is implied by the Greuel-Lossen-Shustin conjecture.

In the talk we consider a smooth (complex) projective surface S, an ample divisor D on S and a divisorial valuation ν of S. As a main result, we show the existence of several equivalent statements to the minimality of ν with respect to D, which provides several equivalent statements to the valuative Nagata conjecture in terms of interesting algebraic and geometric tools.

In particular, we introduce a (previously undefined) Seshadri constant, and relate minimal valuations to this constant, Newton-Okounkov bodies, Zariski chambers and nef divisors of the surfaces determined by the valuations.

Our results provide a **framework** for stating a (valuative) Nagata conjecture for smooth surfaces.