MATH 371
HOMEWORK SET 1
DUE 17.10.2012, WEDNESDAY

Remember: x, y, z, etc. a, b, etc. , A, B, etc. stand for integers! p stand ALWAYS for a prime number

(1) Show that
$$2^{32} + 1 \equiv 0 (\bmod 641).$$
(Hint: Eliminate y from the equation $x^4 + y^4 = x^7 y + 1$ and use $2^4 + 5^4 = 2^7 5 + 1$)

(2) Prove that if $2^k - 1$ is a prime number, then
$$2^{k-1}(2^k - 1)$$
is a perfect number.(Remember: A number n is called perfect if $\sigma(n) - n = n$.)

(3) Prove that if $f(n)$ is multiplicative then
$$g(n) = \sum_{d|n} f(d)$$
is also multiplicative.

(4) Euler's totient or phi function, $\varphi(n)$ is a function that counts the number of positive integers less than or equal to n that are relatively prime to n. For example, $\varphi(6) = 1$ as there is only 5 which is both smaller than 6 and relatively prime to 6.
   i. Compute $\varphi(p)$.
   ii. Show that φ is multiplicative.(Recall that a function f is multiplicative if $f(m)f(n) = f(mn)$ whenever $(m, n) = 1$. Hint: You will need Chinese Remainder Theorem!)
   iii. Find a formula for $\varphi(p^k)$.

(5) Find x and y satisfying
$$54x + 17y = 136$$

(6) If exists, find all solutions of
$$39x + 47y = 4151$$
in positive integers.

(7) Show that if there are integer solutions to the equation
$$ax + by + cz = d$$
then the greatest common divisor of a, b and c divides d

(8) Prove Theorem 5 in your notes.

(9) If x_o, y_o, z_o are integers satisfying
$$x^2 + y^2 + z^2 = 3xyz$$
then show that $x_1 = x_o$, $y_1 = y_o$ and $z_1 = 3x_o y_o - z_o$ are also solutions. Describe in detail how this can be used to find infinitely many solutions in positive integers starting with $x_o = y_o = z_o = 1$.

(10) i. Given a binary quadratic form $f(x, y) = Ax^2 + Bxy + Cy^2$ with Δ is a perfect square describe all solutions of the equation
$$f(x, y) = p.$$
   assuming $(A, B, C) = 1$.
   ii. Explain what happens if $(A, B, C) \neq 1$.
   iii. Use your previous result to find all solutions of the equation
$$2x^2 + 5xy + 2y^2 = 5$$

Notes: You may write your solutions in the language you find appropriate.