MATH 371
HOMEWORK SET 2
DUE 07.11.2012, WEDNESDAY

Remember: x, y, z, etc. a, b, etc., A, B, etc. stand for integers! p stand ALWAYS for a prime number; $\mathbf{Z}$ denotes the set of integers.

(1) Find *all* solutions of the Diophantine equation

$$x^2 + y^2 = 100049$$

(Hint: 100049 is a prime number!)

(2) Find *all* solutions of the Diophantine equation

(3) Prove that $x^2 - 11y^2 = 7$ has no solutions.

(4) Prove that $x^2 - 5y^2 = 1$ has infinitely many solutions.

(5) Let G be a group acting on itself by conjugation (Check your notes for a definition). Describe the orbits of this action if G is abelian.

(6) Let G be any group and consider the action of G on itself (i.e. $\Omega = G$ in our notation). Define maps $G \times \Omega \longrightarrow \Omega = G$

a. $g \cdot_L \omega) := g\omega$, and
b. $g \cdot_R \omega) := \omega g$.
i. Show that both maps are in fact actions of G onto itself (a. is called left action, and b. is called right action).
ii. Compare right action and left action when G is abelian?

(7) Let G be any group and let $\mathrm{Aut}(G)$ denote the *group* of automorphisms of G, that is

$$\mathrm{Aut}(G) := \{\varphi : G \longrightarrow G \mid \varphi \text{ is an isomorphism}\}.$$

i. Show that the map $\cdot : \mathrm{Aut}(G) \times G \longrightarrow G$ sending (φ, g) to $\varphi \cdot g := \varphi(g)$ defines an action of $\mathrm{Aut}(G)$ onto G
ii. For every $g \in G$ show that the homomorphism $\varphi_{g_o} : G \longrightarrow G$ sending g to $\varphi_{g_o}(g) := (g_o^{-1})gg_o$ is an automorphism of G.
iii. Show that the map $\iota : G \longrightarrow \mathrm{Aut}(G)$ sending each $g_o \in G$ to the isomorphism φ_{g_o} is an injective group homomorphism(i.e. a monomorphism.)
iv. Show that the image $\iota(G)$ of G in $\mathrm{Aut}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Terminology: In literature, these groups have special names: the image $\iota(G)$ is called the *inner automorphisms* of G, and denoted by $\mathrm{Inn}(G)$. Since it is a normal subgroup, the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is also a group called the group of *outer automorphisms* of G, denoted by $\mathrm{Out}(G)$.

(8) Let f be a binary quadratic form. Show that the set of automorphisms of f, $\mathrm{Aut}(f)$, is in fact a group.

(9) Show that $U = \begin{pmatrix} 16 & 55 \\ 25 & 56 \end{pmatrix}$ is an automorphism of the form $f = (5, 14, -11)$.

(10) For any given integers x_o and y_o which are relatively prime, show that there are integers x_o' and y_o' such that the matrix

$$\begin{pmatrix} x_o & x_o' \\ y_o & y_o' \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{Z}).$$

Notes: You may write your solutions in the language you find appropriate.

NOTE THE DATE! WE WILL NOT HAVE CLASSES DURING *BAYRAM*.