MATH 371
HOMEWORK SET 3
DUE 21.11.2012, WEDNESDAY

Remember: x, y, z, etc. a, b, etc. , A, B, etc. stand for integers! p stand ALWAYS for a prime number.

(1) Let f be a binary quadratic form. Show that the set of automorphisms of f, $\mathrm{Aut}(f)$, is in fact a group.

(2) Show that $U = \begin{pmatrix} \frac{16}{25} & \frac{55}{56} \end{pmatrix} \begin{pmatrix} 391 & 1155 \\ 630 & 1861 \end{pmatrix}$ is an automorphism of the form $f = (5, 14, -11)$.

(3) Let $f = (A, B, C)$ be an arbitrary binary quadratic form and $U = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{Aut}(f)$. Show that $A|r$.

(4) Show that $\begin{pmatrix} 13 & 24 \\ \underline{172} & 133 \end{pmatrix} \begin{pmatrix} 13 & 24 \\ 72 & 133 \end{pmatrix} \in \mathrm{Aut}(f)$; where $f = (3, 5, -1)$. Can you find any other matrix in $\mathrm{Aut}(f)$?

(5) Let $\mathcal{T} := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \leq \mathrm{PGL}_2(\mathbf{Z})$.

i. Show that $\mathcal{T} = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbf{Z} \right\}$.

ii. For any $t = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \mathcal{T}$ and and binary quadratic form $f = (A.B, C)$ compute $t \cdot f$.

iii. Show that $f = (A, B, C)$ and $f' = (A', B', C')$ are in the same $\mathcal{T}$-orbit (i.e. the *sets* $\mathcal{T} \cdot f$ and $\mathcal{T} \cdot f'$ are equal) if

a. $\Delta(f) = \Delta(f')$

b. $A = A'$

c. $B' = B + 2As$ for some $s \in \mathbf{Z}$

Hint: It is enough to compare the last components! In fact, converse to the above statement is also true. Can you prove?

iv. Write two binary quadratic forms $f = (A, B, C)$ and $f' = (A', B', C')$ with $\Delta(f) = \Delta(f')$ and $A = A'$ but f and f' are not in the same $\mathcal{T}$ orbit.

(6) For any given integers x_o and y_o which are relatively prime, show that there are integers x_o' and y_o' such that the matrix

$$\begin{pmatrix} x_o & x_o' \\ y_o & y_o' \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{Z}).$$

Show that these integers are *not* unique!

(7) Let x and y be two non-zero relatively prime integers. Show that for any $U_o = \begin{pmatrix} x & r_o \\ y & s_o \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{Z})$ we have

$$\left\{ \begin{pmatrix} x & r \\ y & s \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{Z}) \mid r, s \in \mathbf{Z} \right\} = \mathcal{T} \cdot U_o.$$

Show also that the same claim holds even if x and y are not relatively prime.

Notes: You may write your solutions in the language you find appropriate.