MATH 371
HOMEWORK SET 4
DUE 12.12.2012, WEDNESDAY

Remember: a, b, etc. stand for integers! p stand ALWAYS for a prime number. $\mathcal{O}_\Delta$ stands for the ring of integers of the quadratic number field $\mathbf{Q}(\sqrt{\Delta})$. α, β etc. stand for elements of $\mathcal{O}_\Delta$.

(1) Show that the intersection of two distinct quadratic number fields, $\mathbf{Q}(\sqrt{\Delta})$ and $\mathbf{Q}(\sqrt{\Delta'})$ is $\mathbf{Q}$.

(2) For $\alpha \in \mathbf{Q}(\sqrt{\Delta})$, we define the *norm* of α to be the rational number

$$\alpha \cdot \overline{\alpha}.$$

Show that

i. $N(\alpha) = 0 \Leftrightarrow \alpha = 0$.

ii. $N(\alpha\beta) = N(\alpha)N(\beta)$.

iii. If $\alpha \in \mathcal{O}_\Delta$ the $N(\alpha) \in \mathbf{Z}$. However, show the converse to this statement is false by giving three examples of non-integers whose norm is rational integer, i.e. find three elements in the set $\mathbf{Q}(\sqrt{\Delta}) \setminus \mathcal{O}_\Delta$ whose norms are integers.

(3) Show that each of the following numbers are primes in $\mathbf{Q}(\sqrt{-5})$:

i. $3 + 2\sqrt{-5}$,

ii. 37,

iii. $1 + 2\sqrt{-5}$.

(4) Show that 2 and 3 are not primes in $\mathbf{Q}(\sqrt{6})$. (In fact, they both can be written as a product of two associate primes.)

(5) Show that there is no integer in $\mathbf{Q}(\sqrt{7})$ of norm 3 but that 3 is not a prime in $\mathbf{Q}(\sqrt{7})$.(We have seen in class that 2 is a prime in $\mathbf{Q}(\sqrt{-47})$ because there are no elements in $\mathcal{O}_\Delta$ with norm 2.)

(6) Suppose that ε is a unit in $\mathbf{Q}(\sqrt{\Delta})$ and $\sqrt{\varepsilon}$ is an integer(i.e. an element of $\mathcal{O}_\Delta$). Show that $\sqrt{\varepsilon}$ is a unit.

(7) Prove that $\mathcal{O}_\Delta$ is not a UFD for $\Delta =$:

i. -17 (<u>Hint:</u> Try to factor 18 to see that the number of factors in two decompositions may be different.)

ii. -26 (<u>Hint:</u> Try to factor 27 to see that the number of distinct primes appearing in two decompositions may be different.)

(8) Show that 2 and 3 are primes in $\mathbf{Q}(\sqrt{10})$. (<u>Hint:</u> Try to compute their norms. Then reduce modulo 10.)

<u>Notes:</u> You may write your solutions in the language you find appropriate.