MATH 371
HOMEWORK SET 5
DUE 10.01.2013, THURSDAY

Remember: a, b, etc. stand for integers, unless otherwise stated! p stand ALWAYS for a prime number and $\mathfrak{p}$ stand ALWAYS for a prime ideal. $\mathcal{O}_\Delta$ stands for the ring of integers of the quadratic number field $\mathbf{Q}(\sqrt{\Delta})$. α, β etc. stand for elements of $\mathcal{O}_\Delta$.

(1) In class, we've found unit for every $\mathbf{Q}(\Delta)$ with $\Delta < 0$. Let now $\Delta > 0$ and $\varepsilon = a + b\sqrt{\Delta} \in \mathcal{O}_\Delta$; where a and b are allowed to be halves of odd rational integers. Show that if $\varepsilon > 1$ is a unit then $a > 0$ and $b > 0$. Deduce that there is a unique smallest unit greater that 1 in $\sqrt{\Delta}$, denoted by ε_o. Show that every unit of $\mathcal{O}_\Delta$ is of the form $\pm\varepsilon_o^n$ for some $n \in \mathbf{Z}$. Note: This unique unit is called the fundamental unit of $\mathcal{O}_\Delta$.

(2) Let $\mathfrak{A}$ be an ideal of $\mathcal{O}_\Delta$. Show that $\overline{\mathfrak{A}} = \{\overline{\alpha} \mid \alpha \in \mathfrak{A}\}$ is also an ideal of $\mathcal{O}_\Delta$.

(3) Show that norm is multiplicative on ideals, i.e. given two ideals $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{O}_\Delta$ we have $N(\mathfrak{A}\cdot\mathfrak{B}) = N(\mathfrak{A})\cdot N(\mathfrak{B})$

(4) For an ideal $\mathfrak{A}$ of $\mathcal{O}_\Delta$ show that $\mathfrak{A} \cdot \overline{\mathfrak{A}} = (N(\mathfrak{A}))$.

(5) Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_\Delta$. Show that $\mathfrak{p}$ is divisible only by itself and the ideal (1)!

(6) Factor (10) as a product of prime ideals in $\mathbf{Q}(\sqrt{6})$. (Hint: Remember that $\mathcal{O}_\Delta$ for $\Delta = 6$ is not a UFD and mimic the method of the *extended example* we did in class.)

Notes: You may write your solutions in the language you find appropriate.