MATH 468
EXERCISE SET 2

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(1) Let $\alpha_1, \cdots, \alpha_n$ be distinct integers and consider the polynomial
$$f(X) = \left(\Pi_{i=1}^{n} (X - \alpha_i)\right) - 1 \in \mathbf{Q}[X].$$
Show that f is irreducible.
<u>Hint:</u> Assume first that f can be written as a product of two polynomials: $f(X) = g(X)h(X)$, where g and h are monic polynomials of degree $< n$. Compute $g(\alpha_i) + h(\alpha_i)$ for each $i = 1 \cdots, n$.

(2) Prove that $\mathbf{Q}(\sqrt{12}) = \mathbf{Q}(\sqrt{3})$. More generally, let p and q be two distinct integers and consider the fields $K_1 = \mathbf{Q}(\sqrt{p})$ and $K_2 = \mathbf{Q}(\sqrt{q})$. Show that $K_1 = K_2$ if and only if pq is the square of an integer.

(3) Let $K = \mathbf{Q}(\alpha)$ where α satisfies the equation: $\alpha^3 - \alpha^2 + \alpha + 2 = 0$. Express the elements $(\alpha^2 + \alpha + 1)(\alpha - 1)\alpha$ and $(\alpha - 1) - 1$ in terms of the basis $\{1, \alpha, \alpha^2\}$ of $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$.

(4) Using the polynomial $f(X) = X^3 - X + 1$ construct a field, K, with 8 elements. Show that group $(K \setminus \{0\}, \cdot)$ is a cyclic group, i.e. it is generated by one element.

(5) Prove that $\mathbf{Q}(\sqrt{2} + \sqrt{3})$ is equal to $\mathbf{Q}(\sqrt{2}, \sqrt{3})$, and deduce that $[\mathbf{Q}(\sqrt{2} + \sqrt{3}) : \mathbf{Q}] = 4$. Find the minimal polynomial $f_\alpha(X) \in \mathbf{Q}[X]$ of $\alpha = \sqrt{2} + \sqrt{3}$.

(6) Suppose that K/k is a field extension of prime degree, i.e. $[K : k] = p$ for some prime number p. Show that any proper subextension K' of K containing k is k.

(7) Let $\{p_1, \cdots, p_n\}$ be a non-empty set of distinct prime numbers and consider the field $K = \mathbf{Q}(\sqrt{p_1}, \cdots, \sqrt{p_n})$. Show that $\sqrt[3]{p_i} \notin K$ for $i = 1, \cdots, n$.

(8) Let K/k be a field extension and $\alpha \in K$ such that $[k(\alpha) : k] = 2d + 1$ for some $d \geq 1$. Show that $k(\alpha) = k(\alpha^2)$.

(9) Consider the field extension $\mathbf{Q}(\sqrt{-2}, \sqrt{2})/\mathbf{Q}$. Show that this extension has degree 4. This extension contains a subextension, L, i.e. $\mathbf{Q}(\sqrt{-2}, \sqrt{2})$ has a subfield L, other than $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}\sqrt{-2}$, such that $[L : \mathbf{Q}] = 2$. Which field is this?

(10) Let K/k be a field and let R be a **ring** containing k. Show that R is a field.

(11) Determine the splitting field of the polynomial $f(X) = X^4 + 2 \in \mathbf{Q}[X]$.

(12) Find the splitting field of $X^4 - 4$ over $\mathbf{Q}(\sqrt{2})$.

(13) Determine the splitting field of the polynomial $f(X) = X^4 + X^2 + 1 \in \mathbf{Q}[X]$.

(14) Let $f(X) = X^4 + aX^2 + b \in \mathbf{Q}[X]$ be an irreducible polynomial. And let K_f be the splitting field of f. Show that $[K_f : \mathbf{Q}]$ is either 8 or 4. Give examples of each case.