Université Galatasaray, Département de Mathématiques
2012 - Second Semestre – Math 468 - Théorie de Corps
Examen Partiel, 28 mars 2013 – Ayberk Zeytin
90 mins. **Nom & Prénom: Answer Key** ____________

Question:	1	2	3	Total
Points:	40	10	5	55
Score:				

Question1 (40 points)
Prove or disprove the following statements:

i. $K = \mathbb{F}_3[X]/(X^5+1)$ is an extension of $\mathbb{F}_3$.

ii. The field $K = \mathbf{Q}(\sqrt{-1}\sqrt[4]{5})$ is the splitting field of $X^4 - 5$ over $\mathbf{Q}$.

iii. For any field k every $\alpha \in k$ is algebraic over k.

iv. There are irreducible polynomials $f(X) \in \mathbf{Q}[X]$ for which the sum of the two distinct roots is a rational number.

v. There are irreducible polynomials $f(X) \in \mathbf{Q}[X]$ for which the difference between two distinct roots is a rational number.
(Hint: What is wrong with the polynomial $g(X) = f(X) - f(X+q)$?)

vi. Let $f(X) \in \mathbf{Q}[X]$ be any polynomial of degree > 2 and let α, β be two **distinct** roots of $f(X)$. Then $\mathbf{Q}(\alpha) \cong \mathbf{Q}(\beta)$.

vii. Let $f(X) \in \mathbf{Q}[X]$ be an irreducible polynomial of degree > 2 and let α, β be two **distinct** roots of $f(X)$. Then $\mathbf{Q}(\alpha) \cong \mathbf{Q}(\beta)$.

viii. Let p be a prime number and let $\alpha = \sqrt[p]{2} \in \mathbf{R} \subset \mathbf{C}$, then $\mathbf{Q}(\alpha^t) = \mathbf{Q}(\alpha)$ for any $t \in \{1, \cdots, p-1\}$.

Solution:

i. False. The polynomial is not irreducible, hence the quotient is not even a field!

ii. False. K does not contain the real root $\sqrt[4]{5}$.

iii. True. It is a root of the polynomial $X - \alpha \in k[X]$ and thus algebraic.

iv. True. Consider the polynomial $f(X) = X^2 + X + 1$. Its roots are $\omega_1 = \frac{-1+\sqrt{-3}}{2}$ and $\omega_2 = \frac{-1-\sqrt{-3}}{2}$, which are not rational numbers, hence $f(X)$ is irreducible. Their sum $\omega_1 + \omega_2 = \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2} = -1 \in \mathbf{Q}$.

v. False. Suppose that $f(X)$ is an irreducible polynomial having two roots α and $\alpha+q$; where $q \in \mathbf{Q}$ so that their difference is in $\alpha + q - \alpha \in \mathbf{Q}$. Set $g(X) = f(X) - f(X+q)$. Note that $\deg(g) < \deg(f)$, because their leading terms agree. However $g(\alpha) = f(\alpha) - f(\alpha+q) = 0$, i.e. α is a root of g, contradiction to the irreducibility of f.

vi. False. Consider the polynomial $f(X) = (X^3-1)(X^2+1)$. Then the root ζ_3 and $\sqrt{-1}$ generate two fields, namely $\mathbf{Q}(\zeta_3)$ and $\mathbf{Q}(\sqrt{-1})$, which cannot be isomorphic, simply because they have different dimensions as a vector space over $\mathbf{Q}$.

vii. True. Let α and β be two roots of the minimal polynomial $f(X) \in \mathbf{Q}[X]$. As minimal polynomials are unique, the field $\mathbf{Q}(\alpha) \cong \mathbf{Q}[X]/(f(X))$ where the isomorphism is given by evaluation at α map, e_α, and similarly, $\mathbf{Q}(\beta) \cong \mathbf{Q}[X]/(f(X))$ by evaluation at β map, e_β. Therefore $\mathbf{Q}(\alpha) \cong \mathbf{Q}(\beta)$.

viii. True. The inclusion $\mathbf{Q}(\alpha^t) \subset \mathbf{Q}(\alpha)$ is trivial. It is enough to show $\mathbf{Q}(\alpha) \subset \mathbf{Q}(\alpha^t)$, i.e. enough to show $\alpha \in \mathbf{Q}(\alpha^t)$. The greatest common divisor of t and p is 1 hence there are integers n and m so that $tn + pm = 1$. But then,

$$\begin{aligned} \alpha^1 &= \alpha^{tn+pm} \\ &= \alpha^{tn}\alpha^{pm} \\ &= (\alpha^t)^n(\alpha^p)^m \\ &= \underbrace{(\alpha^t)^n}_{\in \mathbf{Q}(\alpha^t)} \quad \underbrace{2^m}_{\in \mathbf{Q}} \Rightarrow \alpha \in \mathbf{Q}(\alpha^t). \end{aligned}$$

Question2 (10 points)

Let p and q be two primes, $p < q$. Set

$$f(X) = x^4 - 2(p+q)X^2 + (p-q)^2 \text{ and } g(X) = X^4 - (p+q)X^2 + pq.$$

i. Show that $f(X)$ is the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over $\mathbf{Q}$.
ii. Find all roots of $f(X)$ in $\mathbf{C}$.
iii. Find the splitting field of $f(X)$ over $\mathbf{Q}$.
iv. Show that the splitting field of f is the same as that of g over $\mathbf{Q}$.

Solution:

i. $f(X)$ is monic, and $f(\sqrt{p} + \sqrt{q}) = 0$. So it is enough to check that f is irreducible.

ii. All the roots of $f(X)$ are $\{\pm\sqrt{p} \pm \sqrt{q}\}$.

iii. We claim that $K_f = \mathbf{Q}(\sqrt{p} + \sqrt{q})$ is the splitting field. In class, we've seen that $K_f = \mathbf{Q}(\sqrt{p}, \sqrt{q})$. K_f contains 3 proper subfields(apart from $\mathbf{Q}$), $\mathbf{Q}(\sqrt{p})$, $\mathbf{Q}(\sqrt{q})$, and $\mathbf{Q}(\sqrt{pq})$. Note that $\sqrt{q} \notin \mathbf{Q}(\sqrt{p})$, and $\sqrt{p} \notin \mathbf{Q}(\sqrt{q})$, hence in either case $\sqrt{p}+\sqrt{q}$ is not an element in the desired subfield. Similarly, $\sqrt{p}+\sqrt{q} \notin \mathbf{Q}(\sqrt{p}, \sqrt{q})$. So $K_f = \mathbf{Q}(\sqrt{p} + \sqrt{q})$ is *the* splitting field of $f(X)$.

iv. All the roots of g are $\{\pm\sqrt{p}, \pm\sqrt{q}\}$. They clearly are elements of K_f. Proceeding as in part (iii.) we deduce the result.

Question3 (5 points)

Which worldwide reputable orchestra will visit İstanbul on the occasion of 20th anniversary of passing of the founder (Dr. Nejat F. Eczacıbaşı) of Istanbul Foundation for Culture and Arts (İKSV)?

Solution:

The New York Philharmonic