Université Galatasaray, Département de Mathématiques
2012 - Second Semestre – Math 468 - Théorie de Corps
Examen Partiel, 07 mai 2013 – Ayberk Zeytin
120 mins. Nom & Prénom: Solutions

Question:	1	2	3	4	Total
Points:	30	15	20	5	70
Score:					

Question1 (30 points)
Determine whether the following statements are true of false. If true prove, if false give a counter-example:

T Let $\alpha \in \mathbb{F}_p$ be arbitrary and let $f(X) = X^p - X + \alpha \in \mathbb{F}_p[X]$. f is separable.

Recall: Thm 13: α is a multiple root of $f(x) \in k[x]$ $\Leftrightarrow (D_x f)(\alpha) = 0$.

Now: $(D_x f)(x) = p \cdot X^{p-1} - 1 = -1 \neq 0 \quad \forall \alpha \in \mathbb{F}_p = k$.

$\Rightarrow$ f does not have any multiple roots.

$\Rightarrow$ f is separable.

F Let $L = \mathbb{F}_{19}(x)$, and for $y = x^{19}$ let $K = \mathbb{F}_{19}(y) = \mathbb{F}_{19}(x^{19})$. The extension L/K is separable.

The minimal polynomial of x over the field $K = \mathbb{F}_{19}(y)$ is:

$$f(t) = t^{19} - y \in \mathbb{F}_{19}(y)[t].$$

But: $f(t) = t^{19} - y = t^{19} - x^{19} = (t-x)^{19}$

$\Rightarrow$ f is not separable.

(OR: $(D_t f)(t) = 19 \cdot t^{18} = 0. \quad \forall t \in \mathbb{F}_{19}(y)$.
$\underset{\text{(Thm 19)}}{\Rightarrow}$ f is not separable)

$\Rightarrow$ L/K is not separable.

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F For an extension K/k and $\alpha, \beta \in K$ there always exist an automorphism $\sigma \in \mathrm{Aut}(K/k)$ with $\sigma(\alpha) = \beta$.

Any automorphism $\sigma : K \to K$ fixes the prime field of K;

hence: for any two distinct elements in the prime field of K

cannot be mapped onto each other.

For instance: there is no $\sigma \in \mathrm{Aut}(\mathbb{F}_2)$ sending $\bar{0}$ to $\bar{1}$.

F Every automorphism of every field leaves fixed an infinite number of elements.

Any element of $\mathrm{Aut}(\mathbb{F}_2)$ fixes only 2 elements

(More generally, if K is a finite field of characteristic p,

any $\sigma \in \mathrm{Aut}(K)$ fixes at least p, at most finitely

many elements (i.e. $|K|$))

T Every automorphism of every field leaves fixed at least 2 elements.

The smallest field is $\mathbb{F}_2$.

The identity fixes $\mathbb{F}_2$ & $|\mathbb{F}_2| = 2$!

T The extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not Galois.

The minimal polynomial of the generator is $X^4 - 2 \in \mathbb{Q}[X]$.

The ~~roots~~ of $X^4 - 2 \in \mathbb{C}$ are:

$$\sqrt[4]{2},\ \underbrace{\zeta_4 \cdot \sqrt[4]{2},\ \zeta_4^2 \sqrt[4]{2},\ \zeta_4^3 \cdot \sqrt[4]{2}}_{\notin \mathbb{Q}(\sqrt[4]{2})}.$$

For any $\sigma \in \mathrm{Aut}(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})$: $\sigma(\sqrt[4]{2})$ is also a root.
But the only root of $X^4 - 2$ in $\mathbb{Q}(\sqrt[4]{2})$ is $\sqrt[4]{2} \Rightarrow \sigma(\sqrt[4]{2}) = \sqrt[4]{2}$.
Any homom. is determined uniquely by its action on the generator $\Rightarrow \sigma = id$.
$\Rightarrow |\mathrm{Aut}(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})| = |\{id\}| = 1 < [\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4 \Rightarrow$ result.

Question2 (15 points)

Show that every finite extension is contained in a Galois extension.

Let K/k be any finite extension. (Thm 17 $\Rightarrow$ splitting fields of separable polynomials are Galois ext^{ns}.) K/k is finite $\Rightarrow$ K/k is algebraic.
$\Rightarrow \exists\ \alpha_1, \dots, \alpha_n \in K$, each α_i is algebraic over k (i.e. a root of a polynomial $p_i(X) \in k[X]$ with $p_i(\alpha_i) = 0$), with the property that

$$k(\alpha_1, \dots, \alpha_n) = K.$$

Let $f_{\alpha_i}(X) = f_i(X)$ be the minimal polynomial of α_i over $k[X]$ & let

$$f(X) = \prod_i f_i(X).$$

& let K_f be the splitting field of the polynomial $f(X) \in k[X]$.
By Theorem 17 K_f/k is Galois & contains $\alpha_1, \dots, \alpha_n \Rightarrow K_f \supseteq K$.

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Question3 (20 points)

Recall that given two subfields, $K_1; K_2$ of a field K their compositum K_1K_2 is defined to be the set of all finite sums of the form $\Sigma_i a_i b_i$ with $a_i \in K_1$ and $b_i \in K_2$. Let K/k be a Galois extension with automorphism group $G = \mathrm{Gal}(K/k)$ and G_1, G_2 be two subgroups of G. Set $K_1 = \mathrm{Fix}G_1$ and $K_2 = \mathrm{Fix}G_2$.

i. Prove that $\mathrm{Aut}(K/K_1K_2) = G_1 \cap G_2$

Let $\sigma \in \mathrm{Aut}(K/K_1K_2)$. Then σ fixes $K_1K_2 \Rightarrow \sigma|_{K_1} = id_{K_1}$ & $\sigma|_{K_2} = id_{K_2}$.

$\Rightarrow \sigma|_{K_1} \in G_1$ & $\sigma|_{K_2} \in G_2 \Rightarrow \sigma \in G_1 \cap G_2 \Rightarrow \underline{\mathrm{Aut}(K/K_1K_2) \subseteq G_1 \cap G_2}$

Conversely, let $\sigma \in G_1 \cap G_2 \Rightarrow \sigma \in G_1$ & $\sigma \in G_2$.

$\Rightarrow \sigma(K_1) = K_1$ & $\sigma(K_2) = K_2$.

Then: for any $\sum_i a_i b_i$: $\sigma(\sum_i a_i b_i) = \sum_i \sigma(a_i)\sigma(b_i) = \sum_i a_i b_i$

$\Rightarrow \sigma$ fixes K_1K_2. $\Rightarrow \sigma \in \mathrm{Aut}(K/K_1K_2) \Rightarrow \underline{G_1 \cap G_2 \subset \mathrm{Aut}(K/K_1K_2)}$

ii. Prove that $\mathrm{Aut}(K/K_1 \cap K_2) = \langle G_1, G_2 \rangle$; where $\langle G_1, G_2 \rangle$ is the group generated by the two groups G_1 and G_2 in G.

Similarly: let $\sigma \in \mathrm{Aut}(K/K_1 \cap K_2)$. Then: $\sigma|_{K_1 \cap K_2} = id|_{K_1 \cap K_2} \Rightarrow \sigma \in \langle G_1, G_2 \rangle$

$\Rightarrow \underline{\mathrm{Aut}(K/K_1 \cap K_2) \subseteq \langle G_1, G_2 \rangle}$.

Conversely, let $g \in \langle G_1, G_2 \rangle$ then: $g(K_1 \cap K_2) = K_1 \cap K_2$ as g is a product of elements of G_1 & G_2 $\Rightarrow g \in \mathrm{Aut}(K/K_1 \cap K_2)$

$\Rightarrow \underline{\langle G_1, G_2 \rangle \subseteq \mathrm{Aut}(K/K_1 \cap K_2)}$

Question4 (5 points)

Bonus Write the name of one movie that has taken part in this years(32nd) International Istanbul Film Festival.