MATH 501
EXERCISES 1

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse.

(1) Show that the eight points $(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3})$ are on S^2 and compute their images under φ.

(2) Can you find 4 points on S^2 which form a tetrahedron? `Hint:` Use (1).

(3) Describe the following sets geometrically:
- $\psi(C_r)$; where $C_r \subset \mathbf{C}$ is the circle with radius r and center 0.
- $\psi(\ell_m)$; where $\ell_m \subset \mathbf{C}$ is the line passing through 0 with slope m
- $\varphi(S^2 \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_3 = 0\})$
- $\varphi(S^2 \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_2 = 0\})$
- $\varphi(S^2 \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x_1 = 0\})$

(4) Recall that for $z = x + iy \in \mathbf{C}$ we define complex conjugation by $\bar{z} = x - iy$. Using stereographic projection complex conjugation can be interpreted as a map from S^2 to S^2. Can you give a geometric description of this interpretation?

(5) Investigate the nature of following functions at ∞ (e.g. are the meromorphic?, do they have poles, zeroes?, if yes of what order?, etc.)
- $f(z) = z + 3$
- $f(z) = z + 1/z$
- $f(z) = e^{z/(z+1)}$
- $f(z) = \sin(1/(z-1))$
- $f(z) = ze^{1/z}$
- $f(z) = \frac{z+1}{z^2}$

(6) Determine the zeroes and poles of the following functions on $\mathbf{P}^1$ together with their multiplicities.
- $f(z) = z^3 + z$
- $f(z) = 3z - 2$
- $f(z) = \frac{z^2+3}{z^4+3z^2+2}$
- $f(z) = z^2 - 2z + 1$
- $f(z) = \frac{2z+1}{z(z^3-5}$

Can you make a conjecture incorporating the multiplicities of zeroes and poles of a function on $\mathbf{P}^1$.