MATH 501
EXERCISES 2

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse.

(1) Find the zeroes and poles of the following functions and verify that $\sum_{a \in \mathbf{P}^1} \nu_a(f) = 0$:

- $f(z) = \frac{z^2+3}{z^4+3z^2+2}$
- $f(z) = \frac{z+2}{z^2+7z+12}$
- $f(z) = \frac{6}{z^3+6z^2+11z+6}$
- $f(z) = \frac{z+2}{z^2+5z}$
- $f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$

(2) Investigate the covering of $\mathbf{P}^1$ determined by the following functions (i.e. find the number of sheets, the branch points and the nature of branching):

- $f(z) = \frac{z^3}{z^4+27}$
- $f(z) = \frac{z^2}{(z-1)^2(z-2)}$
- $f(z) = \frac{z-1}{z^3-z}$
- $f(z) = \frac{z^4}{1-z}$

(3) Let f be a rational function so that if $|z| = 1$ then $|f(z)| = 1$. Show that if α is a zero of $f(z)$ then $\beta = \frac{1}{\alpha}$ is a pole of $f(z)$. Can you guess a general form for $f(z)$?

(4) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be distinct points on $\mathbf{P}^1$ and let $k_1, \ldots, k_n$ and $l_1, \ldots, l_m$ be positive integers. Find necessary and sufficient conditions for the existence of a rational function $f(z)$ with a zero of order k_i at a_i and a pole of order l_j at b_j for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.