MATH 501
EXERCISES 3

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse.

(1) Let X be a set and G a group acting on X. Show that the relation

$$x_1 \sim x_2 :\Leftrightarrow \text{ there is a } g \in G \text{ so that } g \cdot x_1 = x_2,$$

defines and equivalence relation on X.

(2) Let $\Delta \subset \mathbf{P}^1$ be any subset.
- Show that the set of elements of $\mathrm{PSL}_2(\mathbf{C})$ fixing Δ set-wise, denoted by $\mathrm{Stab}G$ is a subgroup.
- For $\Delta = \{0, 1, \infty\}$ show that $\mathrm{Stab}\Delta$ is isomorphic to the symmetric group on 3 letters, S^3.

(3) Show that 3 points on $\mathbf{P}^1$ determine a unique circle in $\mathbf{P}^1$. Deduce that $\mathrm{PSL}_2(\mathbf{C})$ acts transitively on the set of circles in $\mathbf{P}^1$.

(4) For any distinct 4 elements, z_0, z_1, z_2 and z_3 in $\mathbf{P}^1$ we define their *cross ratio* as:

$$[z_0, z_1; z_2, z_3] := T(z_0);$$

where $T \in \mathrm{PSL}_2(\mathbf{C})$ is the unique transformation on $\mathbf{P}^1$ sending z_1 to 0, z_2 to 1 and z_3 to ∞.
- ► Show that $\mathrm{PSL}_2(\mathbf{C})$ does not act 4-transitively on $\mathbf{P}^1$.
- ► Show that there is a element $T \in \mathrm{PSL}_2(\mathbf{C})$ sending and distinct z_0, z_1, z_2 and z_3 to w_0, w_1, w_2 and w_3 (resp.) if and only if $[z_0, z_1; z_2, z_3] = [w_0, w_1; w_2, w_3]$. How unique is T? (`Hint:` Use Exercise 2).
- ► Investigate what happens to the cross-ratio (defined in Exercise 2) if you act on it by the symmetric group on 4 letters.