MATH 501
EXERCISES 4

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse.

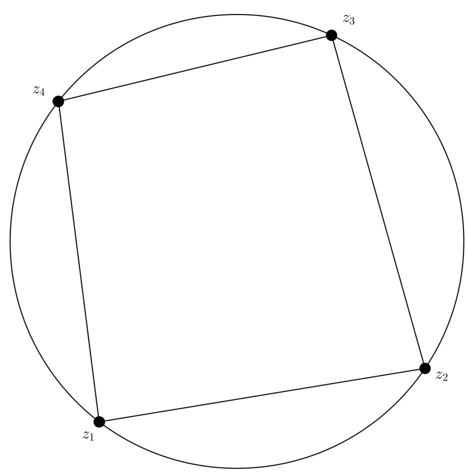
(1) Use Theorem 12 of lecture notes to prove once again that if $T \in \mathrm{PGL}_2(\mathbf{C})$ fixes more that 2 points then $T = I$.

(2) Show that for any two square matrices A and B, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(3) Find $f \in \mathrm{PSL}_2(\mathbf{C})$ so that
- $f(0) = i$, $f(i) = \infty$, $f(\infty) = 1$.
- $f(1) = 1$, $f(0) = i$, $f(\infty) = -i$.
- $f(2) = 5$, $f(i) = 0$, $f(-1) = 1$.

Determine images of the lines $\ell_1 = \{z \in \mathbf{C}\colon \mathrm{im}(()z) = 0\}$ and $\ell_2 := \{z \in \mathbf{C}\colon \mathrm{Re}(z) = 0\}$ under f.

(4) Let $\mathbb{D} := \{z \in \mathbf{C}\colon |z| < 1\}$ be the unit disk and $\mathbb{H} := \{z \in \mathbf{C}\colon \mathrm{im}(()z) > 0\}$ be the upper half plane.
- Find a Möbius transformation f so that $f(0) = -1$, $f(i) = 0$ and $f(\infty) = 1$.
- Show that $f(\mathbb{H}) = \mathbb{D}$. Explain why this must be true geometrically and algebraically.



(5) Let z_1, z_2, z_3 and z_4 be four distinct points on a circle as shown in Figure 4.
- Verify that $[z_1, z_2; z_3, z_4] < 0$ but $[z_1, z_3; z_2, z_4] > 0$.
- Show that $[z_1, z_2; z_3, z_4] + 1 = [z_1, z_3; z_2, z_4] > 0$ and deduce that

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|$$

- Can you express this equality geometrically? (`Hint:` Use a Möbius transformation to map the circle C deterimined by z_i to $\widehat{\mathbf{R}}$)

(6) Let C be a circle in $\mathbf{C}$ with equation

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0.$$

- If $\alpha = 0$ then C becomes a line. In this case, find the slope and x and y intercepts of C in terms of β and γ.
- If $\alpha \neq 0$ the C is an honest circle. In this case, find its centre and radius in terms of α, β and γ.

(7) Let T be a non-identity Möbius transformation. Show that another Möbius transformation S commutes with T if S and T have the same fixed points.

(8) Determine $\mathrm{Stab}(\mathbb{D})$ in $\mathrm{PSL}_2(\mathbf{C})$.

(9) Show that the circle C given by equation $\alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \gamma = 0$; where α and γ are reals, does not intersect the real axis if and only if $(\mathrm{Re}(\beta))^2 < \alpha\gamma$