MATH 501
EXERCISES 5

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse.

(1) Show that hyperbolic elements in $\mathrm{PSL}_2(\mathbf{C})$ do fix a disk whereas loxodromic elements do not.

(2) Show that not every elliptic element in $\mathrm{PSL}_2(\mathbf{C})$ is of finite order.

(3) Show that a non-constant rational function cannot be periodic.

(4) Show that
- $\mathbf{Z}[i] = \{a + bi \in \mathbf{C}\colon a, b \in \mathbf{Z}\}$, and
- $\mathbf{Z}[e^{2\pi i/3} = \{a + b(e^{2\pi i/3}) \in \mathbf{C}\colon a, b \in \mathbf{Z}\}$

are discrete additive subgroups, i.e. lattices, in $\mathbf{C}$ and find their Dirichlet regions. On the other hand, show that the set $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} \in \mathbf{C}\colon a, b \in \mathbf{Z}\}$ is not a discrete subset of $\mathbf{C}$, and hence is not a lattice, although it is a 2 dimensional $\mathbf{Z}$-module! (`Hint:` Observe that $\sqrt{2} - 1 < 1$ and hence arrive to a conclusion by showing that $\mathbf{Z}[\sqrt{2}]$ is dense in $\mathbf{R}$.) Prove a similar result for the sets $\mathbf{Z}[\sqrt{d}]$; where d is a positive square-free integer.

(5) Let $h(z) = \pi^2 \csc^2(\pi z) - \sum_{n=-\infty}^{\infty}$. Prove that:
- $h(z + 1) = h(z)$ for all $z \in \mathbf{C}$.
- $\lim_{z \longrightarrow 0} h(z)$ is finite.
- Deduce that h has finite limit at all integers.
- Show that $h(z)$ is bounded for all $z \in [0, 1] \subset \mathbf{R}$, i.e. there is some $M \in \mathbf{R}_+$ so that $|h(z)| < M$ for any $z \in [0, 1]$. Deduce that $h(z)$ is continuous for any $z \in \mathbf{R}$.
- Show that $h(z/z) + h((z + 1)/2) = 4h(z)$. Use this to show that $M = 0$.
- Deduce that the series expansion for $\csc(\pi z)$.

(6) Prove argument principle using residues, i.e. prove that if f is analytic inside and on a regular closed curve γ, then the number of zeroes of f inside γ is equal to $\frac{1}{2\pi i}\int_\gamma f'/f$. Can you explain why this is called the argument principle?

(7) Prove Rouché's theorem, i.e. prove that if f and g are two analytic functions inside and on a closed curve γ and if $|f(z)| > |g(z)|$ for all $z \in \gamma$, then the number of zeroes of f in the region enclosed by γ is the same as the number of zeroes of $f + g$ inside γ. (`Hint:` First compute $\int_\gamma f'/f$ if f is the product of two functions. Then write $f + g = f(1 + g/f)$. What happens to the resulting integrand?) Use Rouché' theorem to find
- $3e^z - z$ in $|z| \leq 1$
- $1/3\, e^z - z$ in $|z| \leq 1$
- $z^4 - 5z + 1$ in $|z| \leq 2$
- $z^6 - 5z^4 + 3z^2 - 1$ in $|z| \leq 1$

(8) Using Rouché's theorem derive fundamental theorem of algebra.

(9) Determine the residues of the following functions at their poles:
- $f(z) = \frac{1}{z^2 + z^4}$
- $f(z) = \cot(z)$
- $f(z) = \csc(z)$
- $f(z) = ze^{1/z}$
- $f(z) = \frac{1}{z^2 + 3z + 2}$

(10) Using residues evaluate the following integrals:
- $\int_{|z|=1} \cot(z)dz$
- $\int_{|z|=1} \sin(1/z)dz$

- $\int_{|z|=2} ze^{3/z} dz$