MATH 501
EXERCISES 6

A. ZEYTİN

Throughout by $\mathbf{C}$ ($\mathbf{P}^1$, resp.) we denote the field of complex numbers (Riemann sphere, resp.) and z is always a point of $\mathbf{P}^1$. By $\varphi\colon S^2 \setminus \{N = (0,0,1)\} \longrightarrow \mathbf{C}$ we denote the stereographic projection and by ψ its inverse. Ω denotes a lattice in $\mathbf{C}$ generated by two $\mathbf{R}$-linearly independent complex numbers ω_1 and ω_2.

(1) Say $f\colon \mathbf{C} \longrightarrow \mathbf{P}^1$ an elliptic function with periods ω_1 and ω_2 so that $\omega_1/\omega_2 \in \mathbf{R}$.
- ▶ Assume that $\omega_1/\omega_2 \in \mathbf{Q}$. Write $\omega_1/\omega_2 = p/q$ where p and q are relatively prime. Show that the assumption that f being elliptic (i.e. doubly periodic) is equivalent to f being simply periodic with period $\omega_0 = \frac{1}{q}\omega_2$.
- ▶ If ω_1/ω_2 is irrational then f is constant. (`Hint`: Show that whenever τ is irrational the set $\{m - n\tau \colon m, n \in \mathbf{N}\}$ is dense in $\mathbf{R}$).

(2) We define the Weierstraß sigma function as
$$\sigma(z) = z\Pi'_{\omega\in\Omega}(1 - \frac{z}{\omega})\exp(\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2).$$
Assuming the convergence of σ show that
- ▶ σ has a simple zero at each lattice point ω,
- ▶ σ is odd.

Define the Weierstraß zeta function as $\zeta(z) = \frac{\sigma'}{\sigma}(z)$(note that this is NOT the Riemann zeta!).
- ▶ Using the fact that ζ is the *logarithmic derivative* of σ, find an infinite sum which represents ζ.
- ▶ Use previous part to see that ζ is an odd function.
- ▶ Write ζ' as a rational function of $\wp$ and $\wp'$.
- ▶ Show that ζ is NOT elliptic. (`Hint`: Using the above question, integrate to see $\zeta(z+\omega_i) = \zeta(z) + \eta_i$ for $i = 1, 2$. Then, for any $z \in \mathbf{Z}$ let P be the fundamental parallelogram with one vertex at P and compute $\int_{\partial P} \zeta(z)dz$ to see that $2\pi i = \eta_1\omega_1 + \eta_2\omega_2$. Why does this say that ζ is not elliptic?)
- ▶ Use the fact that ζ is not elliptic and is the logarithmic derivative of σ to show $\sigma(z+\omega) = -\sigma(z)\exp(\eta(z + \frac{1}{2}\omega))$ whenever $\frac{1}{2}\omega \notin \Omega$.

(3) Show that any rational function of $\wp$ and $\wp'$, say $\frac{f(\wp,\wp')}{g(\wp,\wp')}$ can be written as $f_1(\wp) + \wp' f_2(\wp)$, where $f_1, f_2 \in \mathbf{C}(\wp)$. (`Hint`: Use the differential equation relating $\wp$ and $\wp'$.)

(4) Let $[a_1], \ldots, [a_r]$ and $[b_1], \ldots, [b_s]$ be distinct points in $\mathbf{C}/\Omega$ and let $k_1, \ldots, k_r$ and $l_1, \ldots, l_s$ be positive integers. If the conditions
- i. $\sum_{i=1}^r k_i = \sum_{j=1}^s l_j$,
- ii. the sets $[a_1] \cup \ldots \cup [a_r]$ and $[b_1] \cup \ldots \cup [b_s]$ are disjoint,
- iii. $\sum_{i=1}^r k_i a_i \equiv \sum_{j=1}^s l_j b_j \mathcal{M}\mathrm{od}(\Omega)$,

hold, then there is an elliptic function f who has zeroes of order k_i at each a_i for $i = 1, 2, \ldots, r$ and poles of order l_j at b_j for each $j = 1, 2, \ldots, s$ and no other zeroes and poles. (`Hint`: Consider the function $f(z) = \frac{\Pi_{i=1}^r\left(\sigma(z-a_i)^{k_i}\right)}{\Pi_{j=1}^s\left(\sigma(z-b_j)^{l_j}\right)}$.)

(5) Show that
- ▶ $\wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 - (\wp(u) + \wp(v))$, and
- ▶ $\wp'(u+v) = \left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)\wp(u+v) + \frac{\wp(u)\wp'(v)-\wp(v)\wp'(u)}{\wp(u)-\wp(v)}$.