Université Galatasaray, Département de Mathématiques
2015 - Fall Semester – Math 501 - Advanced Analysis
Final, 04 Jan. 2016 – Ayberk Zeytin
Take Home **Name & Surname:** ____________________

Question:	1	2	3	Total
Points:	20	36	12	68
Score:				

This exam includes 3 questions. The total number of points is 68.
You are expected to submit your solutions via e-mail to ayberkz@gmail.com before Jan. 08, 2016 Friday at 14h00.
You may talk about the questions among yourselves, you may consult lecture notes and reference books listed on the course syllabus. However, all work submitted has to be yours and yours alone. Any violations will be dealt with according to the relevant code of Galatasaray University.

Question 1 (20 points)
Let f be a rational function from $\mathbf{P}^1$ to $\mathbf{P}^1$ of degree d. By $\nu_\alpha(f)$ we denote the order of f at $\alpha \in \mathbf{P}^1$.

(a) (6 points) Write the general form of a function for which ∞ is neither a pole nor the image of a ramification point with $f(\infty) = 0$.

(b) (6 points) Let f be such a function, i.e. ∞ is neither a pole nor a ramification point and $f(\infty) = 0$. Write $f(z) = f(\alpha) + (z - \alpha)^{e_\alpha(f)} g(z)$; where g is a rational function with $g(\alpha) \neq 0, \infty$. In this setup, compute $v_\alpha(f')$.

(c) (8 points) Deduce that $2d - 2 = \sum_{\alpha \in \mathbf{P}^1} (e_\alpha(f) - 1)$; where d is the degree of f.

Question 2 (36 points)

Let Ω be a lattice in $\mathbf{C}$. Throughout this exercise, we assume that the following function (called the Weierstraß sigma function) is holomorphic on all of $\mathbf{C}$ and the convergence is uniform.

$$\sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} (1 - \frac{z}{\omega}) e^{z/\omega + \frac{1}{2}(z/\omega)^2}$$

(a) (6 points) Show that σ has simples zeroes at each $z \in \Omega$ and no other zeroes.

(b) (6 points) $\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log(\sigma(z)) = -\wp(z)$.

(c) (6 points) For every $\omega \in \Omega$ there are constants $a, b \in \mathbf{C}$, depending on ω so that $\sigma(z+\omega) = e^{az+b}\sigma(z)$.

(d) (6 points) Show that for all $z, a \in \mathbf{C} \setminus \Omega$,

$$\wp(z) - \wp(a) = \frac{\sigma(z+a)\sigma(z-a)}{\sigma(z)^2\sigma(a)^2}.$$

(e) (6 points) Prove that $\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}$, in particular, it is elliptic with respect to Ω.

(f) (6 points) Prove that for any integer n, $\frac{\sigma(nz)}{\sigma(z)^{n^2}}$ is elliptic with respect to Ω.

Question 3 (12 points)

Construct the Riemann surface of the function $f(z) = ((z-1)(z-2)(z-3))^{2/3}$.