Université Galatasaray, Département de Mathématiques
2015 - Fall Semester – Math 501 - Advanced Analysis
Mid-Term 1, 19 Oct. 2015 – Ayberk Zeytin
Take Home **Name & Surname:** ______________________

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	10	12	12	16	12	12	6	90
Score:									

This exam includes 8 questions. The total number of points is 90.
You are expected to submit your solutions via e-mail to ayberkz@gmail.com before Nov. 20, 2015 Friday at 10h00.
I will collect the original solutions on Nov. 23, 2015 Monday.
You may talk about the questions among yourselves, you may consult lecture notes and reference books listed on the course syllabus. However, all work submitted has to be yours and yours alone. Any violations will be dealt with according to the relevant code of Galatasaray University.

Question 1 (10 points)

Let $R_\theta^{(i)}$ be a rotation of S^2 by an angle of θ around x_i axis $(i = 1, 2, 3)$, where

$$S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon \sum_{i=1}^{3} x_i^2 = 1\}.$$

Write the corresponding transformation of $\mathbf{P}^1$ as an element of $\mathrm{PSL}_2(\mathbf{C})$.

Question 2 (10 points)

On $\mathbf{P}^1$, find a rational function with

(a) (4 points) a single zero at -1 and $+1$, and two poles one at -1 and one at ∞.

(b) (6 points) explain why $\mathbf{P}^1$ does not admit any meromorphic function with a double zero at i and a triple pole at ∞ and no other poles and zeroes.

Question 3 (12 points)

Investigate the covering of $\mathbf{P}^1$ by $\mathbf{P}^1$ determined by the function

$$f(z) = \frac{z^3 + 3z^2 - 2z + 1}{z^2(z+1)^2},$$

that is determine the number of sheets, all branching points and the branching structure.

Question 4 (12 points)

Describe the image of the given region under the given transformation:

(a) (4 points) $\mathbb{D} = \{z \in \mathbf{C} \colon |z| < 1\}$ under $T(z) = \frac{iz-i}{z+1}$,

(b) (4 points) $\{z \in \mathbf{C} \colon \mathrm{Re}(z) > 0 \text{ and } \mathrm{im}(z) > 0\}$ under $S(z) = \frac{z-i}{z+i}$,

(c) (4 points) $\{z = x + iy \in \mathbf{C} \colon 0 < x < 1\}$ under $U(z) = \frac{z}{z-1}$.

Question 5 (16 points)

Say p and q are non-zero rational functions. Show that:

(a) (4 points) if $p+q$ is not identically 0 then $\deg(p+q) \leq \max\{\deg(p), \deg(q)\}$,

(b) (4 points) $\deg(p \cdot q) = \deg(p) + \deg(q)$,

(c) (4 points) if f' is not identically 0 then $\deg(f') < \deg(f)$. Deduce that for any natural number n, if $f^{(n)}$ is not identically 0, then $\deg(f^{(n)} \leq \deg(f) - n$.

(d) (4 points) Use above results to prove that $\sqrt[3]{1+z^2}$ and e_z are not rational, and hence are not meromorphic functions on $\mathbf{P}^1$.

Question 6 (12 points)

Let $\mathsf{T}(z) = \frac{4z-5}{2z+3}$.

(a) (4 points) Determine the type of T, i.e. determine whether T is elliptic, hyperbolic, parabolic or elliptic.

(b) (4 points) Determine its fixed points.

(c) (4 points) Find its multiplier.

Question 7 (12 points)

Let $\lambda \in \mathbf{C} \setminus \{0, 1, -1\}$ and set $U_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$.

(a) (8 points) Find two parabolic elements P_1 and P_2 in $\mathrm{PSL}_2(\mathbf{C})$ so that

$$U_\lambda = P_2 \cdot P_1^{-1}.$$

`Hint:` Write the general form of P_1, and write P_2. Try to solve the equation on traces.

(b) (4 points) Deduce that $\mathrm{PSL}_2(\mathbf{C})$ is generated by parabolic elements.

Question 8 (6 points)

Show that any element $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{C})$ of trace 0 is of order two.