MATH 511
EXERCISES 2

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Throughout by K we denote a number field and by $\mathcal{O}_K$ its ring of integers. By R we denote a commutative ring with unity.

(1) Find all conjugates of the following:
- ▶ $\sqrt{2}+\sqrt{3}$
- ▶ $\sqrt[3]{2}$
- ▶ $\zeta_5 = e^{2\pi\sqrt{-1}/5}$

(2) Let $K = \mathbf{Q}(\sqrt{3}, \sqrt{5})$.
- ▶ Describe all embeddings of K into $\mathbf{C}$. (Since $[K : \mathbf{Q}] = 4$, there are 4 such embeddings which will be denoted by σ_1, σ_2, σ_3 and σ_4.)
- ▶ Find an element $\alpha \in K$ so that $\sigma_1(\alpha) = \sigma_2(\alpha) = \sigma_3(\alpha) = \sigma_4(\alpha)$.
- ▶ Find an element $\beta \in K$ so that $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ whenever $i \neq j$.
- ▶ Find an element γ so that $\sigma_1(\gamma) = \sigma_2(\gamma)$, $\sigma_3(\gamma) = \sigma_4(\gamma)$ but $\sigma_2(\gamma) \neq \sigma_3(\gamma)$.
- ▶ For $\theta = \sqrt{3}+\sqrt{5}$, write the map m_θ with respect to the base $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$ and find its norm and trace.
- ▶ Show that $x^4 - 16x^2 + 4$ is the minimal polynomial of $\theta = \sqrt{3}+\sqrt{5}$ and deduce that this is indeed the characteristic polynomial of the map m_θ with respect to the basis $\{1, \theta, \theta^2, \theta^3\}$.
- ▶ Compute $N_K(\theta)$ and $\mathrm{Tr}_K(\theta)$. Observe that $N_K(\theta) = \prod_{i=1}^4 \sigma_i(\theta)$ and $\mathrm{Tr}_K(\theta) = \sum_{i=1}^4 \sigma_i(\theta)$.

(3) Prove that if α is an algebraic number with minimal polynomial $p_\alpha(t) \in \mathbf{Z}[t]$ of degree d, then the characteristic polynomial of m_α is equal to $p_\alpha(t)$. <u>Hint:</u> Try to express m_α with respect to the basis $\{1, \alpha, \alpha^2, \ldots, \alpha^{d-1}\}$. The rest is linear algebra...

(4) Let K be a number field with $[K : \mathbf{Q}] = n$ and let $\sigma_1 \ldots, \sigma_n$ denote the n distinct embeddings of K into $\mathbf{C}$. Show that for any element $\alpha \in K$

$$N_K(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha) \text{ and } \mathrm{Tr}_K(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha)$$

Deduce that if $\alpha \in \mathcal{O}_K$, then both $\mathrm{Tr}_K(\alpha)$ and $N_K(\alpha)$ are integers.

(5) Let α be an algebraic number whose minimal polynomial is of degree n. Compute the discriminant of the set $\{1, \alpha, \ldots, \alpha^{n-1}\}$ in $K = \mathbf{Q}(\alpha)$. <u>Hint:</u> What is a Vandermonde determinant?

(6) Show that for a ring R ascending chain condition is equivalent to the condition that every ideal is finitely generated.

(7) Show that every finite integral domain is a field.

(8) Prove the primitive element theorem. Furthermore, find the element $\alpha \in K = \mathbf{Q}(\sqrt{2}, \sqrt[3]{3})$ for which $K = \mathbf{Q}(\alpha)$.