MATH 511
EXERCISES 3

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Throughout by K we denote a number field and by $\mathcal{O}_K$ its ring of integers. By R we denote a commutative ring with unity.

(1) Prove unicity part of Theorem 9 part (i), i.e. show that if R is a Noetherian domain in which every irreducible is a prime then factorisation in R is unique.

(2) Let $\mathfrak{p}$ be a prime ideal of R. If one considers the map from ideals of $R_{\mathfrak{p}}$ to ideals of R given by $I \mapsto I \cap R$ then, this map is
- ▶ injective, and
- ▶ if restricted to prime ideals, it gives a bijection between prime ideals of $R_{\mathfrak{p}}$ and prime ideals of R contained in $\mathfrak{p}$.

In particular, show that if R is Noetherian, then $R_{\mathfrak{p}}$ is Noetherian.

(3) Let R be an integral domain with field of fractions K, and L some extension of K. Show that R is integrally closed in K if and only if for all $\alpha \in L$ such that α is integral over R, the monic minimal polynomial for α over K has coefficients in R.

(4) Show that if R is a Noetherian ring, and I an ideal, such that there exists a unique prime ideal $\mathfrak{p}$ containing I, then I contains some power of $\mathfrak{p}$. Hint: first show that if an element x is contained in every prime ideal of a ring, then $x^n = 0$ for some n. Do this by considering the family of all ideals such that no power of x lies in them. Next, consider the ring R/I.

(5) Show that if K is the field of fractions of a Noetherian integral domain R, and $I \subset K$ is an R-module (i.e., it is closed under addition, and under multiplication by elements of R), then I is a fractional ideal of R if and only if I is finitely generated.

(6) Show that for any integral domain R, if I and J are fractional ideals of R, then $(I : J)$ is a fractional ideal of R.

(7) Check that for fractional ideals I, J, and a prime ideal $\mathfrak{p}$, we have $I_{\mathfrak{p}} J_{\mathfrak{p}} = (IJ)_{\mathfrak{p}}$, and if R is Noetherian, $(I_{\mathfrak{p}} : J_{\mathfrak{p}}) = (I : J)_{\mathfrak{p}}$.

(8) Set $R = \mathbf{Z}[\sqrt{-3}]$.
- ▶ Show that the ideal (2) in R cannot be written as a product of prime ideals.
- ▶ Show that the ideal $(2, 1 + \sqrt{-3})$ in R does not have any fractional ideal inverse.

Try to find other integers n so that $\mathbf{Z}[\sqrt{n}]$ possesses similar properties.

(9) A geometric version of the above exercise can be given as follows: Set $R = \mathbf{C}[x, y]$ (you may replace $\mathbf{C}$ by any field!).
- ▶ Show that the ideal (x^2, y) in R cannot be written as a product of prime ideals.
- ▶ Show that the ideal (x, y) in R does not have any fractional ideal inverse.