Université Galatasaray, Département de Mathématiques
2016 - Spring Semester – Math 511 - Algebraic Number Theory
Final: Part 1, 29 May 2016 – Ayberk Zeytin
Take Home Name & Surname: ______________________________

Question:	1	2	3	4	5	Total
Points:	2	2	8	2	2	16
Score:						

This exam includes 5 questions. The total number of points is 16.
You are expected to submit your solutions via e-mail to ayberkz@gmail.com before May 30, 2016 Monday, 12h00.
I will collect the original solutions on June 07, 2016 Tuesday.
You may talk about the questions among yourselves, you may consult lecture notes and reference books listed on the course syllabus. However, all work submitted has to be yours and yours alone. Any violations will be dealt with according to the relevant code of Galatasaray University.

Throughout we set K to be a number field and $\mathcal{O}_K$ to be its ring of integers. p will always be a prime number and $\mathfrak{q}$ a prime ideal of $\mathcal{O}_K$. N and Tr stands for norm and trace as usual. When solving the questions the following general theorems might come in handy:
Theorem 1. Let R be a PID and M a finitely generated R-module. Then

$$M \cong (\oplus_{i=1}^{n} R/(x_i)) \oplus R^m$$

for some $m, n \in \mathbf{N} \cup \{0\}$ and $x_i \in R$, $i = 0, 1, \ldots, n$.
Definition. Let K be a number field and p be a prime number. We say a prime ideal $\mathfrak{q}$ lies above the prime number p if $\mathfrak{q}$ occurs in the factorization of p in $\mathcal{O}_K$. The prime number p is said to be ramified in K, or in $\mathcal{O}_K$ if there is at least one prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ which occurs with multiplicity > 1 (i.e. $e_i > 1$ in Equation 1). Else, i.e. if all the prime ideals are not ramified (i.e. $e_i = 1$ for all i in Equation 1) then we say that p totally splits in $\mathcal{O}_K$.
Theorem 2. A prime number p is ramified in $\mathcal{O}_K$ if and only if $p | D_K$.

Question 1 (2 points)
Show that $N_{\mathbf{Q}}(p)^n = p^n = N_K(p\mathcal{O}_K)$. (`Hint:` Remark that $p\mathcal{O}_K$ stands for the ideal generated by p in $\mathcal{O}_K$.)

Question 2 (2 points)
Suppose $[K : \mathbf{Q}] = n$. For the prime number p, say we have the following factorization:

$$p\mathcal{O}_K = \prod_{i=1}^{m} \mathfrak{q}_i^{e_i}; \tag{1}$$

where each $\mathfrak{q}_i$ is a prime ideal of $\mathcal{O}_K$. We set $N(\mathfrak{q}_i) = p^{f_i}$. Show that

$$\sum_{i=1}^{m} e_i f_i = n.$$

Bonus. Relate this to a result that we have verified for rational functions on $\mathbf{P}^1$ last semester.

Question 3 (8 points)
Let $K = \mathbf{Q}(\sqrt{-1})$

(a) (2 points) Find the factorization of (2) in $\mathcal{O}_K$.

(b) (2 points) Find the factorization of (p) in $\mathcal{O}_K$ where p is a prime congruent to 3 modulo 4.

(c) (2 points) Find the factorization of (p) in $\mathcal{O}_K$ where p is a prime congruent to 1 modulo 4.

(d) (2 points) Verify Theorem 2 in each of the above 3 cases.

Question 4 (2 points)
Let n be a square-free positive integer and set $K = \mathbf{Q}(\sqrt{-n})$. For p an *odd* prime not dividing n, show that p totally splits (i.e. $p = \mathfrak{q}_1 \cdot \mathfrak{q}_2$) if and only if $-n$ has a square root modulo p.

Question 5 (2 points)
This time we let $K = \mathbf{Q}(\zeta_q)$, where $\zeta_q = e^{2\pi\sqrt{-1}/q}$ with q being a prime number. Determine all the primes p that split completely in K.