MATH 516
EXERCISES 1

A. ZEYTİN

Throughout we assume the domain U of f is open and connected.

(1) Show that the map $f(z) = z_o + e^{i\theta}z$; where $\theta = \arg(z_o - z_1)$ maps the real axis onto the line ℓ through z_o and z_1.

(2) Find the power series expansion of
- $f(z) = z^3$ around $z_o = 1$
- $f(z) = z^4 - 1$ around $z_o = 2$

(3) Suppose $f\colon U \longrightarrow \mathbf{C}$ is an analytic function and fix some $w \in U$. Let $r > 0$ be a real number so that $D(w, r) \subset U$. Use Cauchy integral formula to derive the power series expansion of f valid at least in $D(w, r)$. Hint: Write $\frac{1}{w-z} = \frac{1}{w(1-\frac{z}{w})}$ and note that $|\frac{z}{w}| < 1$.

(4) Use Liouville's theorem to prove the fundamental theorem of algebra: every non-constant element $p(z) \in \mathbf{C}[z]$ has a zero (or a root) in $\mathbf{C}$. Hint: Assume not and consider $q(z) = \frac{1}{p(z)}$.

(5) Use Exercise 4 and induction on degree to prove that if $p(z)$ is of degree n, then it must have exactly n roots counting multiplicity.

(6) Say f is an entire function so that $f(z + 1) = f(z)$ and $f(z + \sqrt{-1}) = f(z)$ for any $z \in \mathbf{C}$. Show that f is constant. Hint: Show that the image of f has to be bounded.

(7) (Extended Liouville's theorem) Let f be an entire function. If there is an integer $N \geq 0$ and constants A and B so that $|f(z)| \leq A + B|z|^N$ then f is a polynomial of degree at most N. Hint: Use induction on N.

(8) Let f be an entire function. Show that if $\lim_{z\longrightarrow\infty} f(z) = \infty$ then f must be a polynomial. Hint: Observe first that f can have only finitely many zeroes, say $a_1, \ldots, a_n$. What can you say about the function $g(z) = \frac{f(z)}{\prod_{i=1}^{n}(z-a_i)}$

(9) (Minimum principle) Let $f\colon U \longrightarrow \mathbf{C}$ be an analytic function and assume $f(z) \neq 0$ for all $z \in U$. Show that if there is a $z_o \in U$ so that $|f(z_o)| \leq |f(z)|$ for all $z \in U$, then f is constant. Explain, by providing an explicit example, that the non-vanishing of f assumption (i.e. $f(z) \neq 0$ for all $z \in U$) is necessary. Hint: Use maximum principle to an appropriate function.

(10) Deduce fundamental theorem of algebra from Exercise 9, i.e. the minimum principle.

(11) Find the maximum and minimum moduli of $f(z) = z^2 - z$ in $\overline{D(0, 1)}$.

(12) (Schwarz Lemma) Let $f\colon D(0, 1) \longrightarrow D(0, 1)$ be an analytic function fixing the origin, i.e. $f(0) = 0$. Show that:
- $|f(z)| \leq |z|$
- $|f'(0)| \leq 1$

Hint: Apply maximum principle to the function $g(z) = \begin{cases} \frac{f(z)}{z}, \text{ if } z \neq 0 \\ f'(0), \text{ if } z = 0 \end{cases}$.