MATH 516
EXERCISES 2

A. ZEYTİN

Throughout we assume the domain U of f is open and connected.

(1) Determine the Laurent series expansions and the type of singularities of the following functions about the given points:
- ► $f(z) = \dfrac{1}{z^2(z^2-1)}$ around $z_o = 1$,
- ► $f(z) = \dfrac{1}{z^2(z^2-1)}$ around $z_o = 0$,
- ► $f(z) = \cot(z)$ around $z_o = 0$,
- ► $f(z) = \dfrac{e^{1/z^2}}{z+1}$ around $z_o = -1$,
- ► $f(z) = \dfrac{e^{1/z^2}}{z+1}$ around $z_o = 0$,
- ► $f(z) = \dfrac{\sin^3(z^2)}{z^8}$ around $z_o = 0$,

(2) (Casorati Weierstraß Theorem) Suppose $f\colon U \setminus \{z_o\} \longrightarrow \mathbf{C}$ has an isolated essential singularity at the point z_o. Show that the range of f, i.e. $f(D)$, is dense in $\mathbf{C}$ (that is $\overline{f(D)} = \mathbf{C}$). Hint: Assume the contrary so that there is some disk with center w and radius δ with $f(D) \cap B(w, \delta) = \emptyset$. What can you say about the function $\frac{1}{f(z)-w}$?

(3) Say f and g are two functions that have a pole of order m and n at a point z_o, respectively. Decide the orders of poles of the following functions:
- ► $f + g$
- ► fg
- ► f/g

(4) Suppose that f and g are two entire functions so that $|f(z)| < |g(z)|$ for all $z \in \mathbf{C}$. Show that there is some constant $c \in \mathbf{C}$ with $f(z) = cg(z)$ for all $z \in \mathbf{C}$. Hint: After elimininating the case $g \equiv 0$, consider the functions $h = f/g$. Why are singularities of h removable?

(5) Find the Laurent expansion of $\frac{1}{(z-1)z(z+1)}$ valid in:
- ► $0 < |z| < 1$
- ► $1 < |z| < \infty$

(6) Let f be a non-constant entire function. Show that the image of f is dense in $\mathbf{C}$. Hint: Eliminate the polynomial case. Then consider the function $f(1/z)$.

(7) (Rouché's Theorem) Let f and g be two functions analytic in U, simply connnected and let γ be a simple closed curve in U. If $f(z) > g(z)$ for all $z \in \gamma$ then show that the number of zeroes of the function $f + g$ inside the region enclosed by γ is exactly equal to that of f. Hint: First show that if $f = AB$ then $\frac{f}{f'} = \frac{A}{A'} + \frac{B}{B'}$. Apply this to $f + g = f(1 + \frac{g}{f})$. What can you say about the number of zeroes of $1 + \frac{g}{f}$?

(8) Use Rouché'e theorem to find the number of zeroes of the following functions within the indicated regions:
- ► $f(z) = 3e^z - z$ in $|z| \leq 1$
- ► $f(z) = z^4 - 5z + 1$ in $|z| \leq 2$

(9) Evaluate the following integrals assuming the curves are positively oriented:
- ► $\displaystyle\int_{|z|=1/2} \frac{1}{z^2+5z+4}dz$
- ► $\displaystyle\int_{|z|=3/2} \frac{1}{z^2+5z+4}dz$

- $\displaystyle\int_{|z|=516} \frac{1}{z^2+5z+4}\,dz$
- $\displaystyle\int_{|z|=1} \tan(z)\,dz$
- $\displaystyle\int_{|z|=2} \tan(z)\,dz$
- $\displaystyle\int_{|z|=1} \sin(1/z)\,dz$

(10) Suppose f has a simple pole at a point $z_o \in \mathbf{C}$.

$$\mathrm{Res}(f, z_o) = \lim_{z \longrightarrow z_o} (z - z_o) f(z)$$

(11) Show that if f has a pole of order k at z_o then

$$\mathrm{Res}(f, z_o) = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z - z_o)^k f(z) \right)$$

(12) Show that

- $\mathbf{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} \colon a, b \in \mathbf{Z}\}$ is a discrete subgroup of $\mathbf{Z}$.
- $\mathbf{Z}[e^{2\pi\sqrt{-1}/3}] = \{a + be^{2\pi\sqrt{-1}/3} \colon a, b \in \mathbf{Z}\}$ is a discrete subgroup of $\mathbf{Z}$.
- $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} \colon a, b \in \mathbf{Z}\}$ is not a discrete subgroup of $\mathbf{Z}$.

Can you show, more generally, that $\mathbf{Z}[\sqrt{d}] = \{a + b\sqrt{d} \colon a, b \in \mathbf{Z}\}$ is not a discrete subgroup of $\mathbf{Z}$; where $d > 0$ and square-free?