MATH 516
EXERCISES 3

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Throughout we assume the domain U of f is open and connected.

(1) Let Ω be the lattice generated by 1 and $e^{2\pi\sqrt{-1}/3}$. Find a fundamental region for Ω in the shape of an equilateral hexagon. Show that the sum of residues of any function elliptic with respect to Λ is 0.

(2) Show that the order of an elliptic function is independent of the choice of a fundamental parallelogram.

(3) Fix a lattice $\Omega = \Omega(\omega_1, \omega_2)$. Show that the set of functions elliptic with respect to Ω is a field over $\mathbf{C}$ where addition and multiplication is defined pointwise.

(4) Show that if f is elliptic with respect to $\Omega = \Omega(\omega_1, \omega_2)$, then f' is also elliptic with respect to Ω.

(5) Given a meromorphic function with Laurent series $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n(z - z_o)^n$ around the point z_o, the *analytic* (resp. *principal*) *part* of f around z_o is defined as the series $\sum_{n=0}^{\infty} a_n(z - z_o)^n$ (resp. $\sum_{n=-\infty}^{-1} a_n(z - z_o)^n$). Suppose that f and f are functions elliptic with respect to $\Omega = \Omega(\omega_1, \omega_2)$, with
- poles at the same points in $\mathbf{C}$, and
- same principal parts at these poles.

Show that $f(z) - g(z) = C$ for some constant $C \in \mathbf{C}$.

(6) Let f and g be two meromorphic functions elliptic with respect to $\Omega = \Omega(\omega_1, \omega_2)$ with zeroes and poles of the same orders at the same points in $\mathbf{C}$. Show that $f(z) = Dg(z)$ for some constant $D \in \mathbf{C}$.

(7) (The Weierstraß σ-function) For a lattice $\Omega = \Omega(\omega_1, \omega_2)$, we define the *Weierstraß* σ-function as:
$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega\in\Omega\setminus\{0\}} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.$$
Show that
- ▶ Show that $\sigma(z)$ is an odd holomorphic function on $\mathbf{C}$ with simple zeroes at each point of the lattice Ω.
- ▶ Show that $\dfrac{d^2}{dz^2}(\log(\sigma(z))) = -\wp(z)$.
- ▶ Show that there are constants $a, b \in \mathbf{C}$ so that $\sigma(z + \omega) = e^{az+b}\sigma(z)$ for all $z \in \mathbf{C}$. Such a function is called a ϑ-function with respect to Ω.
- ▶ Prove that for any $a \in \mathbf{C} \setminus \Omega$ we have:
$$\wp(z) - \wp(a) = -\frac{\sigma(z + a)\sigma(z - a)}{\sigma(z)^2\sigma(a)^2}.$$
- ▶ Show that:
$$\wp'(z) = -\frac{\sigma(2z)}{\sigma(z)^4}.$$

(8) (The Weierstraß ζ-function) For a lattice $\Omega = \Omega(\omega_1, \omega_2)$, we define the *Weierstraß* ζ-function as:
$$\zeta(z) = \zeta_\Omega(z) = \frac{d}{dz}(\log(\sigma(z))).$$
- ▶ Show that $\zeta(z) = \dfrac{1}{z} + \displaystyle\sum_{\omega\in\Omega\setminus\{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$.
- ▶ Find a linear function, $\eta(z)$, so that $\zeta(z + \omega) = \zeta(z) + \eta(\omega)$.
- ▶ For the function η you found in the previous part, show that $\eta(\omega) = 2\zeta(\omega/2)$ for any $\omega \in \Omega$.
- ▶ Show that $\sigma(z + \omega) = \pm e^{\eta(\omega)(z + \frac{\omega}{2})}\sigma(z)$.