MATH 516
EXERCISES 4

A. ZEYTİN

(1) Recall that
$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2(n+1)} z^{2n};$$
where G_{2n} is the *Eisenstein series*, namely $G_{2n} = \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{\omega^{2n}}$. Define $g_2 = 60G_4$ and $g_3 = 140G_6$[1]. Show that
$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$
Deduce that $K(\Omega) = \mathbf{C}(X)[Y]/(Y^2 - 4X^3 + g_2X + g_3)$.

(2) For $\Omega = \Omega(\omega_1, \omega_2)$, let $e_1 = \wp(\omega_1/2), e_2 = \wp(\omega_2/2)$ and $e_3 = \wp((\omega_1 + \omega_2)/2)$ denote its half-periods. Show that:
- ▶ $e_1 + e_2 + e_3 = 0$,
- ▶ $g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$,
- ▶ $g_3 = 4e_1e_2e_3$.

Deduce from the above discussion that $g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2$ and hence $e_i \neq e_j$, $i \neq j$ if and only if $\Delta(\Omega) = g_2^3 - 27g_3^2 \neq 0$. Hint: Compare the coefficients of the two equations for $\wp'^2$.

(3) For any integer $k \geq 4$
- ▶ Show that
$$(2k+1)(2k-1)(k-3)G_{2k} = 3\sum_{j=2}^{k-2} (2j-1)(2k-2j-1)G_{2j}G_{2k-2j};$$
where G_k is the k^{th} Eisenstein series. Hint: Show first that $\wp''(z) = 6(\wp(z))^2 - 30G_4$ and recall: $G_{2k+1} = 0$ for $k \geq 1$ integer.
- ▶ Show that $G_8 = \frac{3}{7}G_4^2$, $G_{10} = \frac{5}{11}G_4G_6$.
- ▶ Compute G_{12} in terms of G_4 and G_6.

(4) We say that two lattices $\Omega = \Omega(\omega_1, \omega_2)$ and $\Omega' = \Omega(\omega_1, \omega_2)$ are equivalent if there is a non-zero complex number a so that $\Omega' = a\Omega$. Show that this is indeed an equivalence relation.

(5) Let $\Omega = \Omega(1, \tau)$ be any lattice in $\mathbf{C}$ and $M\Omega$ be the lattice $\Omega(\alpha\tau + \beta, \gamma\tau + \delta)$ where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z})$. Finally, let $G_{2n}(\Omega)$ and $G_{2n}(M\Omega)$ be the corresponding Eisenstein series. Show that
$$G_{2n}(M\Omega) = (\gamma\tau + \delta)^{2n} G_{2n}(\Omega).$$

[1] The numbers g_2 and g_3 are called *Weierstraß invariants* of the lattice Ω.