MATH 516
EXERCISES 5

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We fix the following matrices throughout the course:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(1) Write the matrix $\begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix}$ as a product of
- ▶ S and T,
- ▶ S and L,
- ▶ T and L.

Show that the modular group $\mathrm{PSL}_2(\mathbf{Z})$ is generated by any two of the matrices S, L and T. Hint: Note that it is enough to prove only the first item. The rest is a corollary of it. To show the first, prove that any matrix of the form $\begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$ belongs to the subgroup generated by S and T. Then show that by multiplying matrix with a proper sequence os S and T an arbitrary matrix can be put into this form.

(2) Show that $G_{2k}(\frac{pz+q}{rz+s}) = (rz+s)^{2k} G_{2k}(z)$ for any matrix $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{Z})$.

(3) For the lattice $\Omega = \Omega(1, \sqrt{-1})$, show that:
- ▶ $g_2(\Omega) = g_2(\sqrt{-1}) \in \mathbf{R} \setminus \{0\}$,
- ▶ $g_3(\Omega) = g_3(\sqrt{-1}) = 0$

(4) For the lattice $\Omega = \Omega(1, e^{2\pi\sqrt{-1}/3})$, show that:
- ▶ $g_2(\Omega) = g_3(e^{2\pi\sqrt{-1}/3}) = 0$,
- ▶ $g_3(\Omega) = g_3(e^{2\pi\sqrt{-1}/3}) \in \mathbf{R} \setminus \{0\}$

(5) Determine a point τ in the fundamental region, F, of the action of the modular group on $\mathbb{H}$ which is equivalent to:
- ▶ $\frac{1+\sqrt{-1}}{4}$
- ▶ $\frac{1-\sqrt{-1}}{4}$
- ▶ $\frac{516+2\sqrt{-1}}{8}$
- ▶ $\frac{516-2\sqrt{-1}}{8}$