MATH 516
EXERCISES 6

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We fix the following elements of the modular group $\mathrm{PSL}_2(\mathbf{Z})$ throughout the course:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(1) Let f be a meromorphic function with an isolated non-essential singularity at $z = 0$ and no other in $\mathbb{D} = \{z \in \mathbf{C} : |z| < 1\}$. Show that f can have only finitely many zeroes (if any) in $\mathbb{D}$.

(2) Find all elements of $\mathrm{PSL}_2(\mathbf{Z})$ which fix $\sqrt{-1}$, i.e. determine all elements $M \in \mathrm{PSL}_2(\mathbf{Z})$ so that $M \cdot z = z$. Show that they form a subgroup of the modular group of order 2, denoted by $\mathrm{Stab}(\sqrt{-1})$. Similarly, find all elements of the modular group which fix $\zeta_3 = e^{2\pi\sqrt{-1}/3}$. Show that they form a subgroup of order 3, denoted by $\mathrm{Stab}(\zeta_3)$. Show, more generally, that if a group G acts on a set X, then, elements of G which fix an element $x \in X$, denoted $\mathrm{Stab}(x)$, is a subgroup of G.

(3) Fix some $k \in \mathbf{N}$. Let M_{2k} denote the set of modular forms of weight $2k$. Show that M_{2k} is a vector space over $\mathbf{C}$.

(4) Show that the derivative of a modular function is a modular form of weight 2. What can you say about the derivative of a modular form of weight $2k$?

(5) Suppose f and g are modular forms of weight $2k$. Show that $f'g - g'f$ is a modular form. Find its weight.

(6) Recall that a cusp form is a modular form, say f, so that $\lim_{\tau \longrightarrow \sqrt{-1}\infty} f(\tau) = 0$, or, equivalently, its q-series expansion is holomorphic and has 0 constant term. Show that,
   - ▶ the sum and product of two cusp forms is again a cusp form,
   - ▶ the product of a cusp form with a modular function is again a cusp form.

(7) Fix some $N \in \mathbf{N}$. Show that if a holomorphic function $f : \mathbf{C} \longrightarrow \mathbf{C}$ satisfies $f(z) = f(z + N)$, then it can be expressed as a function of the variable $q_N = e^{2\pi\sqrt{-1}z/N}$; that is there is some $\widetilde{f}$ so that $f(z) = \widetilde{f}(q_N)$.