MATH 516
EXERCISES 7

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Recall that:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(1) For a meromorphic function f let $\{f\}$ denote the Schwarzian derivative of f. Explicitly:

$$\{f, z\} = 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2.$$

- Show that $\{f, z\} = \left(\frac{1}{f'}\right)^2 \left(2f'f''' - 3(f'')^2\right)$
- Show that if f is a modular function then $\{f, z\}$ is a modular form of weight 4. Is an analoguous result still valid if f is replaced by a modular form of weight $2k$.
- Show that $\{z, f\}$ is again a modular function whenever f is a modular function.
- Check whether $\frac{\Delta}{\Delta'}$ is a modular form. If yes, what is its weight?

(2) Show that $S \cdot \zeta_3 = \zeta_3 + 1$ and compare $g_2(\zeta_3)$ and $g_2(S \cdot \zeta_3)$ and $g_2(T \cdot \zeta_3)$. Deduce that $g_2(\zeta_3) = 0$. Use similar method to show that $g_3(\sqrt{-1}) = 0$, and $j(\sqrt{-1})(1728)$.

(3) Show that the set of cusp forms $\mathbb{S} = \cup_{k \in \mathbf{N}, k \text{ even}} \mathbb{S}_k$ is a graded ring.

Happy new year!!!