MATH 532
EXERCISES 1

A. ZEYTİN

Exercises marked with $*$ are your homework.

(1) Let I and J are two ideals of a ring R. Decide whether the followings are ideals of R:
- $* \, I + J = \left\{ \sum_{\text{finite}} \alpha_i + \beta_i \colon \alpha_i \in I,\ \beta_i \in I \text{ for all } i \right\}$
- $* \, IJ = \left\{ \sum_{\text{finite}} \alpha_i \beta_i \colon \alpha_i \in I,\ \beta_i \in I \text{ for all } i \right\}$
- $I \setminus J$
- $\sqrt{I} = \{\alpha \in R \colon \alpha^n \in I \text{ for some } n \in \mathbf{N}\}$[1]
- $I \cap J$
- $I \cup J$

(2) Let S be any subset of a ring R. Show that the set:

$$(S) = \left\{ \sum_{\text{finite}} \alpha_i s_i \colon \alpha_i \in R,\ s_i \in S \right\}$$

is an ideal of R.

(3) Let X be an arbitrary subset of $\mathbb{A}^n(k)$. Show that the set

$$I(X) = \{F \in k[X_1, \ldots, X_n] \colon F(x) = 0 \text{ for all } x \in X\}$$

is an ideal of $k[X_1, \ldots, X_n]$.

(4) Verify the following properties of algebraic sets:
- If $\{I_i\}$ is any collection of algebraic sets then $V(\bigcup_{i \in I} I_i) = \bigcap_{i \in I} V(I_i)$.
- If $I \subseteq J$ then $V(I) \supseteq V(J)$.
- For any two polynomials $F, G \in k[X_1, \ldots, X_n]$, $V(FG) = V(F) \cup V(G)$.
- $V(0) = \mathbb{A}^n(k)$ and $V(1) =$.

Deduce that declaring open sets as complements of algebraic sets defines a topology on $\mathbb{A}^n(k)$. This topology is called the Zariski topology on $\mathbb{A}^n(k)$.

(5) $*$ Let $\varphi \colon R \longrightarrow S$ a ring homomorphism and J is a prime ideal of S. Show that $\varphi^{-1}(J)$ is a prime ideal of R. Show, by an example, that the analogous result for maximal ideals is not true.

(6) The following questions are assigned from our textbook which is available at

http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf

1.3, 1.5$*$, 1.6, 1.11, 1.13, 1.15, 1.16$*$, 1,19, 1.24, 1.25, 1.26, 1.29$*$, 1.41$*$, 1.45

[1] An ideal J of R which is the radical of another ideal, i.e. for which there is another ideal I so that $J = \sqrt{I}$, is called a radical ideal.