MATH 532
EXERCISES 2

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Exercises marked with $*$ are your homework.

(1) Decide whether the following algebraic sets are algebraic varieties or not:
- ▶ $V(X^2+Y^2-1, X^2-Z^2-1)$,
- ▶ $V(Y^2-X(X-1)(X-\lambda))$; where $\lambda \in \mathbf{C}$, constant,
- ▶ $V(Y^2-XY-X^2Y+X^3)$,
- ▶ *$\mathbb{A}^n(k)$; where k is a *finite* field - if you are uncomfortable with finite fields, just take $k=\mathbf{Z}/p\mathbf{Z}$; where p is a prime number.

(2) Show, by an explicit example, that Nullstellensatz is false whenever k is not algebraically closed.

(3) *Show that $V=V(Y^2+X^2(X-1)^2) \subset \mathbb{A}^2(\mathbf{R})$ is not an algebraic variety even though $F(X,Y)=Y^2+X^2(X-1)^2 \in \mathbf{R}[X,Y]$ is an irreducible polynomial.

(4) Let I be an ideal in $k[X_1,\dots,X_n]$, set $V=V(I)$. Let q be an arbirtary element of $\mathbb{A}^n(k)\setminus V$.
- ▶ Show that there is a polynomial $F \in k[X_1,\dots,X_n]$ so that $F(p)=0$ for all $p\in V$ and $F(q)=1$.
- ▶ Deduce that if $p_1 \dots, p_r$ are arbirtary distinct points in $\mathbb{A}^n(k)\setminus V$, then, there are polynomials $F_1,\dots,F_r \in k[X_1,\dots,X_n]$ so that
$$F_i(p_j)=\delta_{i,j}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$
<u>Hint:</u> Use previous part of the exercise with $I=I(V\cup\{p_1,\dots,p_{i-1},p_{i+1},\dots,p_r\})$ and all but one point. Why $V\cup\{p_1,\dots,p_{i-1},p_{i+1},\dots,p_r\}$ is algebraic?
- ▶ Show that $V=V(I)$ is a finite set if and only if $k[X_1,\dots,X_n]/I$ is a finite dimensional vector space over k.
- ▶ * Do all the computations explicitly for $k=\mathbf{C}$, $V=\{p_1,p_2,p_3,p_4\}$; where $p_1=\sqrt{-1}, p_2=-\sqrt{-1}, p_3=1, p_4=-1$. That is, find polynomials $F_i(X)\in\mathbf{C}[X]$ so that:
$$F_i(p_j)=\delta_{i,j}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$
for $i,j\in\{1,2,3,4\}$. Determine $I(V)$ explicitly, and show that the ring $\mathbf{C}[X]/I(V)$ is indeed a vector space of dimension 4.

(5) Show that for
- ▶ *$I=(Y^2-YX^2, XY-X^3)\leq k[X,Y]$, $\sqrt{I}=(Y-X^2)$.
- ▶ $I=(X^2,Y^2)\leq k[X,Y]$, $\sqrt{I}=(X,Y)$.
- ▶ $I=(X^2+1)\leq \mathbf{R}[X]$, $\sqrt{I}=I$. Note that $V(I)=\emptyset$!

(6) Let k be a finite field. Show that $\mathbb{A}^n(k)$ is reducible. On the other hand, if k is an infinite field show that $\mathbb{A}^n(k)$ is irreducible.

(7) In this exercise, we assume k is an algebraically closed field unless otherwise stated. Show that the following varieties, V and W, are isomorphic both by establishing explicit polynomial map from V to W or vice versa and by writing an explicit ring isomorphism between their coordinate rings :
- ▶ $V=V(Y-X^2)\subset\mathbb{A}^2(k)$ and $W=\mathbb{A}^1(k)$
- ▶ $V=V(X^2+Y^2-1)\subset\mathbb{A}^2(k)$ and $W=V(XY-1)\subset\mathbb{A}^2(k)$. Decide whether any of these is isomorphic to $\mathbb{A}^1(k)$.
- ▶ $V=V=(Y-X^2,Z-X^3)\subset\mathbb{A}^2(\mathbf{R})$ and $W=V((Y-X^2)^2+(Z-X^3)^2\subset\mathbb{A}^2(\mathbf{R})$

(8) The following questions are assigned from our textbook which is available at

http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf

1.2*, 1.22*, 1.36*, 1.37, 1.38*, 2.2, 2.5, 2.10, 2.11, 2.12, 2.18(to solve this one you'll need to solve 2.2, too.)