MATH 532
EXERCISES 2

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Exercises marked with $*$ are your homework.

(1) Say $f \in R[X_1, \ldots, X_n]$ be a polynomial. As we have seen in class $f = F_0 + F_1 + \ldots + F_d$; where $d = \deg(f)$, and each F_i consists exactly of monomials of degree i in f. A polynomial $f \in R[X_1, \ldots, X_n]$ for which we have $f = F_i$ for some fixed i is called a *homogeneous polynomial* (or a *form*). For instance, $f(X, Y) = X^2 + 2XY + XY^2 + 5 = (5) + (0) + (X^2 + 2XY) + (XY^2)$ is not a form, whereas $G(X, Y) = XY^2 + 3X^2Y + Y^3$ is. Given any $f = F_0 + F_1 + \ldots + F_d \in R[X_1, \ldots, X_n]$, the polynomial $f_* = X_{n+1}^d F_0 + X_{n+1}^{d-1} F_1 + \ldots, X_n F_{d-1} + F_d \in R[X_1, \ldots, X_n, X_{n+1}]$ is called the *homogenization* of f. Similarly, for any homogeneous polynomial $F \in R[X_1, \ldots, X_{n+1}]$, we define $F_* = F(X_1, \ldots, X_n, 1)$. The polynomial F_* is called the *de-homogenization* of F.

- ▶ For $G(X, Y) = XY^2 + 3X^2Y + Y^3$ compute G_*.
- ▶ For $f(X, Y) = X^2 + 2XY + XY^2 + 5 = (5) + (0) + (X^2 + 2XY) + (XY^2)$, compute f^*
- ▶ Show that f_* is a form of degree d.
- ▶ What is the degree of F_*? Under what conditions is F_* a form?
- ▶ Is $\cdot^*\colon R[X_1, \ldots, X_n] \longrightarrow R[X_1, \ldots, X_n, X_{n+1}]$ a ring homomorphism? Is it surejctive?, injective?
- ▶ Is $\cdot_*\colon R[X_1, \ldots, X_n, X_{n+1}] \longrightarrow R[X_1, \ldots, X_n]$ a ring homomorphism? Is it surejctive?, injective?
- ▶ Show that if $F(X_1, \ldots, X_n)$ is any homogeneous polynomial, say of degree d, then for any non-zero $t \in k$, we have $F(tX_1, \ldots, tX_n) = t^d F(X_1, \ldots, X_n)$.
- ▶ Show that $\displaystyle\sum_{i=1}^{n} X_i \frac{\partial F}{\partial X_i} = dF$; where F is assumed to be a homogeneous polynomial of degree d. This is called Euler's lemma.
- ▶ Extend Euler's lemma (previous part) to second partials : $d(d-1)F = \sum_{i,i} X_i X_j \frac{\partial^2 F}{\partial X_i \partial X_j}$.

(2) Two ideals I and J of a ring R is said to be *co-maximal* if $I + J = R$.

- ▶ Show that if I and J are two distinct maximal ideals, then I and J are co-maximal.
- ▶ If I and J are co-maximal, then $IJ = I \cap J$.
- ▶ If I and J are co-maximal, then $R = I \cap J^2$. Remark that $J^2 \subset J$!
- ▶ If I and J are co-maximal, then I^m and J^n are also co-maximal.
- ▶ Two ideals I and J in $k[X_1, \ldots, X_n]$; where k is algebraically closed, are co-maximal if and only if $V(I) \cap V(J) = \emptyset$.

(3) Let $F, G \in k[X_1, \ldots, X_n]$. We say that F is equivalent to G, and write $F \sim G$, is there is a $\lambda \in k \setminus \{0\}$ so that $F = \lambda G$. Show that $\sim$ is an equivalence relation.

(4) Consider the curve $F(X, Y) = X^3 - XY + Y^2 - 1 \in k[X, Y]$. Decide whether the point $(1, 1)$ is simple or multiple. Find the tangent line (or lines) of F at $(1, 1)$ If multiple, compute its multiplicity. Determine (if any) all the multiple points of F.

(5) *Find the simple and singular (non-simple) points of the curve $F(X, Y) = Y^2 - X^2 + X^4$. Draw F in $\mathbb{A}^2(\mathbf{R})$. Guess the tangent line from you drawing at $(\pm 1, 0)$. Then find the tangent line at $(\pm 1, 0)$ algebraically.

(6) For λ and μ fixed but arbitrary constants in $\mathbf{C}$, consider the curve $F_{\lambda,\mu}(X, Y) = \mu(X^3 + Y^3 + 1) + 3\lambda XY$[1]. Determine λ_o and μ_o so that F_{λ_o,μ_o} is singular. Determine the multiplicity at the singular points.

(7) * Find the singular points of the curve $F(X, Y) = X^2Y^2 + Y^2 + X^2$ and determine their multiplicities. How many points are there of this curve in $\mathbb{A}^2(\mathbf{R})$? How many points of this curve are there in $\mathbb{A}^2(\mathbf{C})$? What is its tangent line at $(0, 0)$?

(8) The following questions are assigned from our textbook which is available at

http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf

[1] This is called the family of Steiner cubics.

2.20, 2.21, 2.23, 2.25, 2.26, 2.28[2]*, 2.29, 2.33, 2.43*, 2.44*, 2.46, 2,48, 2.49, 2.54, 2.55. 2.56, 3.2, 3.3, 3.4*

[2]The order function in this question is termed as *valuation* in modern literature. A ring with a valuation is called a valuation ring.