MATH 504
EXERCISES 1

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Unless otherwise stated G is a group.

(1) For each $n \in \mathbf{N}$, the symmetric group $\mathfrak{S}_n$ is the group of permutations of the set $\{1, 2, \ldots, n\}$.
  - Convince yourself that $\mathfrak{S}_n$ has order $n!$.
  - Show that $\mathfrak{S}_n$ is not abelian for $n \geq 3$.
  - More generally, show that $C_{\mathfrak{S}_n} = \{(1)\}$.
  - Recall that elements of $\mathfrak{S}_n$ are denoted by products of cycles. Given $\sigma = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10)$, determine $i = 1, 2, \ldots, 10$ for which σ^i is again a cycle and for which it is not.
  - Let σ, τ be two arbitrary cycles. Show that $\sigma\tau = \tau\sigma$ if two cycles are disjoint.
  - The *length* of a cycle is defined to be the number of integers that appear in the cycle. A *transposition* is defined as a cycle of length 2. As any element of $\mathfrak{S}_n$ is a product of cycles, every element $\sigma \in \mathfrak{S}_n$ can be written as a product of finitely many transpositions. The *sign* of the cycle σ is $\mathrm{sign}(\sigma) := (-1)^l$ where l is the number of transpositions in σ. Show first that the parity of l (i.e. l being even or odd) is well defined.
  - Determine the signs of following elements of $\mathfrak{S}_n$:
    - $(1\,5\,6\,8\,3\,4)(2\,4)(3\,2)$
    - $(1\,6\,3)(7\,2\,5\,6\,1)$
    - $(1\,3)(1\,5\,7)(1\,3\,8)(7\,8)$
    - $(1\,3)(1\,5\,7)(1\,3\,8)(7\,8) \circ (1\,6\,3)(7\,2\,5\,6\,1)$
  - An element $\sigma \in \mathfrak{S}_n$ is called even (resp. odd) $\mathrm{sign}(\sigma) = 1$ (resp. $\mathrm{sign}(\sigma) = -1$). Show that $\mathrm{sign}\colon \mathfrak{S}_n \to \{\pm 1\}$ is a group homomorphism.

(2) Find all subgroups of the group
  - $\mathbf{Z}/24\mathbf{Z}$,
  - $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$,
  - $\mathfrak{S}_3$.

(3) Show that if a group has only finitely many subgroups, then the group itself must be a finite group.

(4)
  - For H and K two subgroups of G show that $H \cap K$ is also a subgroup of G.
  - Let $\mathcal{A}$ be a non-empty collection of subgroups of G. Show that

$$\bigcap_{H \in \mathcal{A}} H$$

    is a subgroup of G.
  - Let A be any *subset* of G. The subgroup generated by A, denoted $\langle A \rangle$, is defined as :

$$\langle A \rangle := \bigcap_{H \leq G,\, A \subseteq H} H.$$

    Deduce, using previous part of the exercise that $\langle A \rangle$ is a subgroup of G.
  - Let B be any *subset* of G. Define

$$\widetilde{B} := \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} | n \in \mathbf{N},\ \alpha_i = \pm 1 \text{ with } a_i \text{ not necessarily distinct}\}.$$

    Show that $\widetilde{B}$ is a subgroup of G.
  - Show finally that for any subset A of a group G, we have $\langle A \rangle = \widetilde{A}$.
  - Show that if $A \subseteq B$ then $\langle A \rangle \subseteq \langle B \rangle$.
  - Show that if $H \leq G$ then $\langle H \rangle = H$.
  - When $A = \{g_1, \ldots g_n\}$ is a finite subset of G, we denote $\langle A \rangle$ by $\langle g_1, \ldots, g_n \rangle$. In particular, when $A = \{g\}$. we denote $\langle A \rangle = \langle g \rangle$. In $\mathfrak{S}_4$, determine $\langle \{(1\,2), (2\,3\,4)\} \rangle$
  - Compute $\langle (1\,2\,3) \rangle$ in $\mathfrak{S}_4$.
  - Show that $H \cup K$ is in general not a subgroup of G.

(5) A subgroup H of G for which there is some $g \in H$ so that $H = \langle g \rangle$ is called a *cyclic subgroup*. In particular, if there is some element $g \in G$ so that $G = \langle g \rangle$, then G itself is called a *cyclic group*.

- Show that $\mathbf{Z}$ is a cyclic group.
- Decide whether the group $\mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ is cyclic.
- Decide whether the group $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$ is cyclic.
- How many generators does $\mathbf{Z}/24\mathbf{Z}$ have?
- More generally, how many generators does $\mathbf{Z}/n\mathbf{Z}$ have, where $n \in \mathbf{N}$?
- Let G, G' be two groups. Show that if G is cyclic, say generated by $g_o \in G$ (i.e. $G = \langle g_o \rangle$), then any group homomorphism $\varphi\colon G \to G'$ is completely determined by $\varphi(g_o)$. That is to say, if $\psi\colon G \to G'$ is another group homomorphism, then $\varphi = \psi$ if and only if $\varphi(a) = \psi(a)$.
- Using the previous exercise find all automorphisms of $\mathbf{Z}/7\mathbf{Z}$, $\mathbf{Z}/9\mathbf{Z}$. What can you say about all the automorphisms of $\mathbf{Z}/n\mathbf{Z}$, for $n \in \mathbf{N}$?
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(6) Let G be a group and $g \in G$ be an arbitrary element. We define the *order* of g, denoted by $\mathrm{ord}(g)$, to be the size of the subgroup generated by g, that is $\mathrm{ord}(g) = |\langle g \rangle|$ (infinity is allowed!).

- Find the order of $(1\,2\,4\,3\,5\,6\,8\,7) \in \mathfrak{S}_8$.
- Find the order of $(1\,2\,4\,3)(5\,6\,8\,7) \in \mathfrak{S}_8$.
- Find the order of $(1\,2\,4)(\,3\,5\,6\,8\,7) \in \mathfrak{S}_8$.
- In $\mathfrak{S}_5$ determine all elements of order 4.
- For an element $x \in G$ show that x and x^{-1} have the same orders.
- Prove that for any $x, g \in G$ we have $\mathrm{ord}(xgx^{-1}) = \mathrm{ord}(g)$.
- Show, using the preovious part, that for any $x, y \in G$ we have $\mathrm{ord}(xy) = \mathrm{ord}(yx)$.
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(7) Let G be a group having a unique element $g \in G$ of order 2. Show that for all $x \in G$ we have $gx = xg$, that is $g \in C_G$. Hint: For any $x \in G$ compute $(xgx^{-1})^2$.

(8) Let G be a group which satisfies :

$$\text{for all } g, g' \in G \text{ one has } (gg')^2 = g^2 g'^2$$

Show that G is abelian.

(9) Let G be a group which satisfies :

$$\text{for all } g, g' \in G \text{ one has } (gg')^3 = g^3 g'^3$$

Show that G is abelian.

(10) Find all subgroups of $\mathfrak{S}_4$. Which of these are normal in $\mathfrak{S}_4$?

(11) Let G be a group.

- Fix an arbitrary element $g \in G$. Define $C_G(g) := \{g \in G | xg = gx \text{ for each } x \in G\}$. Show that $C_G(g)$ is a subgroup of G. This subgroup is called the *centralizer* of g in G.
- Let A be a subset of G. Show that the set (called the centralizer of A in G)
$$C_G(A) := \{x \in G \,|\, xg = gx \text{ for all } g \in A\}$$
is a subgroup of G.
- Let A be a subset of G. Define $N_G(A) := \{x \in G \,|\, xA = Ax\}$. Show that $N_G(A)$ is a subgroup of G. This subgroup is called the normalizer of A in G.
- What is the relationship between $C_G(A)$ and $N_G(A)$ if $|A| = 1$?
- Compute $N_G(G)$.
- The group $C_G(G)$ is called the *center* of G and denoted usually by Z_G of C_G. Compute $C_{\mathfrak{S}_4}$.
- Let $G = \mathfrak{S}_4$ and $A = \{(1\,2), (1\,3)\}$. Compute $C_G(A)$ and $N_G(A)$.

(12) Let $\varphi\colon G \to H$ be a group homomorphism between G and H. Show that :

- $\varphi(1_G) = 1_H$
- $\varphi(g^{-1}) = \varphi(g)^{-1}$
- $\varphi(g^n) = (\varphi(g))^n$

(13) Let $\varphi\colon \mathbf{R}^2 \longrightarrow \mathbf{R}$ be the group homomorphism defined by $\varphi(x, y) = x + y$.

- Is φ a group homomorphism?
- Find $\ker(\varphi)$.
- Describe $\varphi(\mathbf{R}^2)$.

(14) Let $\varphi\colon \mathbf{R}^\times \longrightarrow \mathbf{R}^\times$ be the group homomorphism defined by $\varphi(x) = |x|$.

- Is φ a group homomorphism?
- Find $\ker(\varphi)$.
- Describe $\varphi(\mathbf{R}^\times)$.

(15) Let $\varphi\colon \mathbf{C}^\times \longrightarrow \mathbf{R}^\times$ be the group homomorphism defined by $\varphi(x+iy) = x^2+y^2$.

- Is φ a group homomorphism?
- Find $\ker(\varphi)$.
- Describe $\varphi(\mathbf{C}^\times)$.

(16) Let $G = (\mathbf{R}, +)$ and $S^1 := \{z \in \mathbf{C} \mid |z| = 1\}$.

- Show that S^1 is a subgroup of $\mathbf{C}^\times$.
- Define $\varphi\colon G \longrightarrow H$ by $\varphi(x) := e^{2\pi\sqrt{-1}x}$. Show that φ is a group homomorphism.
- Find $\ker(\varphi)$.
- Describe $\varphi(\mathbf{R}^\times)$.