MATH 504
EXERCISES 2

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Unless otherwise stated G is a group.

(1) Show that a subgroup H of G is normal if and only if left cosets of H are at the same time right cosets of H, that is $gH = Hg$ for all $g \in G$.

(2)
- ▶ Show that an intersection of normal subgroups is again a normal subgroup of G. Deduce that it makes sense to talk about the smallest normal subgroup containing a given subset S of G.
- ▶ For any $a, b \in G$, let S be the set of all elements of G of the form $aba^{-1}b^{-1}$. Let C be the smallest normal subgroup containing S[1]. Show that G/C is an abelian group.

(3) Let H be a normal subgroup of G. Show that the canonical map

$$\begin{aligned} \pi_H \colon G &\to G/H \\ g &\mapsto gH \end{aligned}$$

is a group homomorphism whose kernel is exactly H. Show using this that if $\varphi\colon G \to G'$ is a group homomorphism with kernel H, then $G/H \cong \varphi(G)$. This is the first isomorphism theorem.

(4) This exercise will lead to the proof of the second isomorphism theorem. Let H and K be two subgroups of G.
- ▶ Give an example of a group and two subgroups so that HK is not a subgroup of G.
- ▶ Show that if K is a normal subgroup of G, then $HK = KH$. Show further that HK is a subgroup of G.
- ▶ Show that if both H and K are normal subgroups of G, then HK is a normal subgroup of G.
- ▶ Let H, K be two subgroups of a group G, and suppose that K is normal in G. Show that $HK/K \cong H/(H\cap K)$.
  <u>Hint:</u> Restrict the canonical map $G \to G/K$ to H.

(5) Consider the following map :

$$\begin{aligned} \bullet \colon \mathbf{R} \times \mathbf{R}^2 &\to \mathbf{R}^2 \\ (r, (x, y)) &\mapsto (x, rx + y) \end{aligned}$$

Show that $\bullet$ defines an action of $\mathbf{R}$ on $\mathbf{R}^2$. Determine the the equivalence class and stabilizers of $(0, 0)$, $(1, 0)$ and $(1, 1)$

(6) Let G be the group of symmetries of the unit cube in $\mathbf{R}^3$.
- ▶ Show that G is isomorphic to $\mathfrak{S}_4$ by considering its action on the set of pairs of *opposite* vertices of the cube.
- ▶ Show that the action of G on the set of pairs of opposite faces (there are 3 of them) is not faithful[2].

(7) Let X be the polynomial ring in 4 independent variables with integral coefficients, that is $X = \mathbf{Z}[x_1, x_2, x_3, x_4]$. Consider the map :

$$\begin{aligned} \cdot \bullet \cdot \colon \mathfrak{S}_4 \times X &\to X \\ (\sigma, p(x_1, x_2, x_3, x_4)) &\mapsto \sigma \bullet p(x_1, x_2, x_3, x_4) := p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(x_3)}, x_{\sigma(x_4)}) \end{aligned}$$

- ▶ Show that this defines an action of $\mathfrak{S}_4$ onto X.
- ▶ Compute the stabilizer of $x_1 + x_2$ under this action.
- ▶ Find all polynomials that are equivalent to $x_1 + x_2$, i.e. find $[x_1 + x_2]$.
- ▶ Compute the stabilizer of $x_1x_2 + x_3x_4$.
- ▶ Find all polynomials that are equivalent to $x_1x_2 + x_3x_4$, i.e. find $[x_1x_2 + x_3x_4]$.

(8) Let G be a group acting on a set X. We say that the action of G on X is transitive if $|X/G| = 1$, that is there is only one equivalence class. Consider now the action of $G = GL_2(\mathbf{R})$ on $\mathbf{R}^2$ defined by $M \cdot (x, y) \mapsto M(x\ y)^t$.

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[1] This subgroup is called the **commutator subgroup** of G.

[2] Recall that an action is called faithful if the kernel of the associated permutation representation is trivial.

- Show that this action is not transitive on $\mathbf{R}^2$.
- Show on the other hand that it is transitive on $\mathbf{R}^2 \setminus \{0\}$.

(9) Decide whether the action of a group G on itself by conjugation is transitive.

(10) Prove that if a finite group G acts on a set X the $|X| \mid |G|$.

(11) Describe the permutation representation associated with the action $\mathfrak{S}_3$ on itself by multiplication from left.

(12) Consider the quaternion group Q_8. Show that Q_8 is isomorphic to a subgroup of $\mathfrak{S}_8$.

(13) Let $\sigma, \tau \in \mathfrak{S}_n$ be arbitrary. If σ has cycle decomposition

$$\sigma = (a_1\, a_2\, \cdots\, a_{k_1})(b_1\, b_2\, \ldots\, b_{k_2})\cdots$$

then show that the cycle decomposition of $\tau\sigma\tau^{-1}$ is :

$$\tau\sigma\tau^{-1} = (\tau(a_1)\,\tau(a_2)\,\cdots\,\tau(a_{k_1}))(\tau(b_1)\,\tau(b_2)\,\ldots\,\tau(b_{k_2}))\cdots$$

Deduce that two elements of $\mathfrak{S}_n$ are conjugate if and only if they have the same cycle type, hence the number of conjugacy classes in $\mathfrak{S}_n$ is exactly the number of partitions of n.

(14) For the following elements, determine whether they are conjugate of not and if yes, find an explicit element conjugating one to the other :

- $\sigma_1 = (1\,2)(3\,4\,5)$, $\sigma_2 = (1\,2\,3)(4\,5)$
- $\sigma_1 = (1\,2)(3\,4)(5\,6\,7)$, $\sigma_2 = (7\,6)(5\,4)(3\,2)$
- $\sigma_1 = (1\,2)(3\,4)(5\,6)$, $\sigma_2 = (1\,2\,3)(4\,5\,6)$

(15) Determine all finite groups having exactly two conjugacy classes.

(16) Let G be a group acting on a set X. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of X.